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FOUNDATIONS

Quine, Willard Van Orman. **Methods of Logic.** Henry Holt & Company, New York, N. Y., 1950. xxi+264 pp. $2.90.

This is a textbook, suitable for a first course in logic for general students, but it could also serve as an introduction to logic for students who wish to pursue the subject further. The author maintains a high standard of precision and clarity, showing the reader by the text itself that one can discuss logical problems without being mystical, or even obscure. The work is divided into four parts. Of these, part I is concerned mainly with (two-valued) classical sentential calculus; part II is devoted to syllogistic reasoning or rather, more generally, to the study of the one-place predicate calculus; and part III takes up the predicate calculus in general (i.e., without the restriction to one-place predicates). The final part, which is entitled "Glimpses Beyond," deals with singular terms, descriptions, and identity, and, in a more sketchy way, with set theory and the construction of classical mathematics. The transition from each of these parts to the next is made smooth, and a motivation is provided for the next part, by considering, in each case, arguments which cannot be formulated in the systems previously considered. Thus part II begins by pointing out that the categorical syllogism cannot be shown to be valid by the methods of sentential calculus; and the step from the one-place predicate calculus to the general predicate calculus is shown to be desirable by pointing out that within the one-place calculus we cannot infer from the premise "All circles are figures" the conclusion "All who draw circles draw figures." The author examines very carefully some of the problems which arise in connection with the analysis of statements and arguments expressed in nonformal language, making some reasonable suggestions, and pointing out pitfalls that are to be avoided.

Although the book is written in an extremely exact way, the author is concerned, not with trying to make a professional logician of the general student, but with helping him to clarify his own mind, and to learn to apply the results of logic to other fields. Thus the sentential calculus is not presented as a deductive system with rules and primitive formulas, but is given, almost from the first, by means of a decision method. Similarly, a decision method is derived for the one-place predicate calculus, and the importance of having a decision method is appropriately emphasized. Finally, the general predicate calculus is made to rest on the "natural" method of deduction of Gentzen and Jaśkowski; and here the author does not hesitate to complicate the rules of deduction, in order to obtain a system in which the more commonly useful formulas have short and easy derivations. This is not to deny, of course, that the more usual formulations of these systems can be useful in general logical investigations, or that they have interest in themselves; but the formulations here presented appear to be better for the general student in that they are easier to grasp, and facilitate the application of logic to other domains.

J. C. C. McKinsey (Santa Monica, Calif.).

Schütte, Kurt. Schlussweisen-Kalküle der Prädikatenlogik. Math. Ann. **122**, 47–65 (1950).

Gentzen's Hauptsatz [Math. Z. **39**, 176–210, 405–431 (1934)] establishes the eliminability of one of the rules of inference for his calculus of sequences, a deductive structure of greater complexity than the classical predicate calculus. Here the author establishes a similar elimination theorem for a more limited formalization of the classical predicate calculus and certain extensions of it. The only formal entities of the system are formulas built up using alternation, $\vee$, negation, $\bar{\ }$, and quantification. The theorem states that the rule $A \vee B,\ \bar{B} \vee C/A \vee C$ is superfluous. An analogous treatment of the intuitionistic predicate calculus is also presented. The author considers a second formalization of classical predicate calculus in which substitution schemata are made to play as important a role as possible. A normal form for proof in this system is described. *D. Nelson.*

Goodman, Nelson. An improvement in the theory of simplicity. J. Symbolic Logic **14**, 228–229 (1950).

In this paper on the assignment of complexity values to bases of predicates, the author strengthens the cardinal rule stated in an earlier paper [same vol., 32–41 (1949); these Rev. **10**, 668]; it appears that by modifying slightly the original method of computation, this stronger condition can be satisfied as well. *E. W. Beth* (Amsterdam).

Piaget, Jean. Sur la logique des propositions. Arch. Sci. Soc. Phys. Hist. Nat. Genève **3**, 159–161 (1950).

The author arranges the formulas of sentential calculus into a matrix, and makes some rather obvious remarks about them. *J. C. C. McKinsey* (Santa Monica, Calif.).

Myhill, John R. A reduction in the number of primitive ideas of arithmetic. J. Symbolic Logic **15**, 130 (1950).

Let '$\phi(x, y)$' mean, where 'x' and 'y' are variables ranging over positive integers, that there is an integer z such that x is the successor of $y \cdot z$. The author notes that divisibility and the successor relation are both definable in terms of 'ϕ,' presupposing truth functions and quantification over positive integer variables. He further notes that if to 'ϕ' we add what is essentially the Hilbert μ-function as an additional primitive, both the Sheffer stroke function and quantification are definable. Possibly some further reduction is also possible. *R. M. Martin* (Philadelphia, Pa.).

Goodstein, R. L. The formal structure of a denumerable system. Trans. Amer. Math. Soc. **68**, 174–182 (1950).

A denumerable system is one whose fundamental domain consists of a denumerable number of elements. The Hilbert-Bernays system Z_μ is such a system [Grundlagen der Mathematik, vol. 2, Springer, Berlin, 1939, pp. 293 ff.]. A detailed construction of a good deal of classical function theory on the basis of what is essentially the system Z_μ is presented in this paper. The treatment is not couched in logistic symbols but is rather in the manner of a clearly written elementary

mathematics text. The topics taken up successively include signed integers, integer functions, rational numbers, rational functions, and endless decimals. A notation for decimal functions is introduced only metamathematically, and such properties as continuity, uniform continuity, and differentiability are definable only as metamathematical attributes. Hence as metamathematical theorems about the denumerable system, one can prove analogues of the basic theorems of classical analysis. Goodstein's methods are in some respects comparable to those employed in recent papers by Fitch [J. Symbolic Logic **13**, 95–106 (1948); **14**, 9–15 (1949); **15**, 17–24 (1950); these Rev. **9**, 559; **10**, 669; **12**, 2].

R. M. Martin (Philadelphia, Pa.).

Martin, R. M. On virtual classes and real numbers. J. Symbolic Logic **15**, 131–134 (1950).

Previous papers by the author [same J. **8**, 1–23 (1943); **14**, 27–31 (1949); these Rev. **4**, 182; **10**, 668] have dealt with a certain "nominalistic" first order functional calculus, in which quantification is allowed only over variables whose values are individuals. Notations for virtual classes and relations were there introduced, and a theory of general recursive functions of positive integers developed. He now shows that certain portions of the theory of real numbers may be constructed in this system, using the device of virtual classes and relations. However, only denumerable totalities of real numbers may be treated, and quantification is allowable only over denumerable classes. Even the elementary laws of real arithmetic cannot be derived in full generality, and the continuity principle is not provable.

O. Frink (State College, Pa.).

von Bertalanffy, Ludwig. An outline of general system theory. British J. Philos. Sci. **1**, 134–165 (1950).

The author outlines the development of the system concept in diverse sciences and the isomorphism of the laws governing systems. From the general properties of systems he shows how various specific properties result, such as progressive segregation, centralisation, competition, and equifinality (the independence of a final state from initial conditions). Open systems are emphasized. It is concluded that many concepts which have often been considered as anthropomorphic, metaphysical, or vitalistic are consequences of the general properties of systems.

C. C. Torrance (Annapolis, Md.).

ALGEBRA

Carr, Russell E. Enantiomorphism in mathematical models of organic molecules. Iowa State Coll. J. Sci. **24**, 141–188 (1950).

On connait les travaux de Cayley sur les arbres et sur les isomères des composés chimiques. Après avoir rappelé les résultats de Pólya [Acta Math. **68**, 145–254 (1937)], et de Allen et Diehl [même J. **16**, 161–171 (1942); ces Rev. **3**, 259], l'auteur définit deux catégories de stéréoisomères, (1) les non-enantiomorphes, dont la formule développée peut être superposée à son image dans le miroir, (2) les enantiomorphes, dont le schema n'est pas superposable à son symétrique par rapport à un plan. Il se propose d'adapter à l'énumération des stéréoisomères des deux classes la méthode de Pólya, qui n'est pas directement applicable à ce cas. Les investigations sont limitées aux alcools, composés aliphatiques (alkenes) et paraffines; mais la méthode resterait valable dans d'autres cas.

Soit $s(x)$, $i(x)$, $m(x)$ les fonctions génératrices pour le nombre des alcools stéréoisomères, non-enantiomorphes, et enantiomorphes, respectivement. On a: $i(x)=1+xi(x)s(x^2)$; $m(x)=s(x)-i(x)$. Pour les alkenes, en désignant les fonctions analogues par: $a(x)$, $b(x)$, $c(x)$, on a:

$$b(x)=\tfrac{1}{4}x^3[i(x)^4+3s(x^2)^2]$$

et $c(x)=a(x)-b(x)$, les $s(x)$ étant données par la méthode de Pólya. Pour les paraffines, les résultats sont moins simples. La partie importante du travail est la détermination des formules asymptotiques. Soit $\sigma=0.30422$ et $b=1.228$; alors, pour les alcools, en appelant S_n, I_n, et M_n les nombres de stéréoisomères, de non-enantiomorphes, et d'enantiomorphes, on a: $2S_n\sim\sigma^{-n}n^{-\frac{3}{2}}b\pi^{-\frac{1}{2}}$, $I_n\sim\sigma^{-\frac{1}{2}n}n^{-\frac{3}{2}}b^{-1}(2\pi\sigma)^{-\frac{1}{2}}$, $M_n\sim S_n$. Pour les olefines, en désignant les mêmes nombres par A_n, B_n, C_n respectivement, l'auteur trouve: $2A_n\sim\sigma^{-n}n^{-\frac{3}{2}}b\sigma^{\frac{1}{2}}\pi^{-\frac{1}{2}}$, $B_n\sim\sigma^{-\frac{1}{2}n}n(16\sigma b^4)^{-1}$, $C_n\sim A_n$. Des formules analogues sont données pour les paraffines. La dernière partie contient les tables des 20 premières valeurs de $S, I, M, A, B, \cdots$. La comparaison, pour $n=20$, des valeurs approchées avec les valeurs exactes fait apparaître une erreur relative, qui dans certains cas, ne dépasse pas 0.0004.

A. Sade.

Del Vecchio, E. Una proprietà del determinante generale di potenze che si connette con la teoria della dipendenza statistica. Giorn. Mat. Finanz. (3) **8**, 20–42 (1950).

The author studies the expansion of the determinant $|x_j^{k_i}|$, where $j=1, \cdots, m$ is a subscript and k_i, $i=1, \cdots, m$ is a set of positive increasing integral powers of the x_j.

A. M. Mood (Santa Monica, Calif.).

Parodi, Maurice. Quelques propriétés des matrices *H*. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 22–25 (1950).

If the real numbers a_{ij}, $i, j=1, \cdots, n$, satisfy $a_{ii}>\sum_{j\neq i}|a_{ij}|$, then the two roots of $\|a_{ij}\|$ of greatest absolute value must be greater than or equal to min $[a_{ii}-|a_{jk}|]$, where $j\neq k$ and $i, j, k=1, \cdots, n$. The method involves a careful use of the relation: (sum of the absolute values of the roots) $\geqq\sum_i a_{ii}$. In three places $i\neq j$ should read $j\neq k$.

W. Givens.

Kondô, Koiti. A remark to Toeplitz's theorem on normal matrix. Kōdai Math. Sem. Rep. **1950**, 56 (1950).

The Toeplitz theorem that $AA^*=A^*A$, where A^* is the conjugate transpose of A, implies that the elementary divisors of A are simple is proved by writing

$$(xE-A)^{-1}=C(x-\lambda)^{-a}+D(x-\lambda)^{-a+1}+\cdots$$

(λ an eigenvalue of A, $C\neq0$, $a>0$), comparing the derivative and the square of this expression, and using $C^2\neq0$. That $C^2\neq0$ follows from $CC^*=C^*C$.

W. Givens.

Lepage, Th. Sur un théorème de Kronecker relatif aux matrices symétriques. Bull. Soc. Math. Belgique **2** (1948–1949), 26–32 (1950).

The author uses the theory of the Grassman exterior algebra [see Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 288–299, 527–541 (1947); these Rev. **9**, 265] to prove Runge's theorem that all linear relations among the minors of a symmetric matrix are linearly dependent on those of Kronecker [see MacDuffee, The Theory of Matrices, Springer, Berlin, 1933, section 9]. The author applies his theory to prove: If A and B are two matrices of order n such that the n by $2n$ matrix (A, B) is of rank n, there exists a symmetric orthog-

onal matrix S whose elements are 1, -1, or 0 such that $A+BS$ is nonsingular. *B. W. Jones* (Boulder, Colo.).

Richter, H. Über Matrixfunktionen. Math. Ann. **122**, 16–34 (1950).

Let $\mathfrak{B}_0$ be an open domain of matrices A having distinct latent roots, such that $CAC^{-1}\epsilon\mathfrak{B}_0$ if $A\epsilon\mathfrak{B}_0$. Suppose that for each $A\epsilon\mathfrak{B}_0$ a matrix B, of the same order as A, is defined whose elements are single valued continuous functions of the elements of A (written $B=Fu(A)$), then B may be called a general continuous matrix function of A in $\mathfrak{B}_0$ if $B=Fu(A)$ implies $CBC^{-1}=Fu(CAC^{-1})$ for all nonsingular C. The domain of definition of such a matrix function can be extended to $\mathfrak{B}$ including boundary points of $\mathfrak{B}_0$ if a continuous function $A(t)$ of a real parameter t (or a convergent sequence $A(t_n)$ with $t_n\to 0$) can be found such that $A(t)\epsilon\mathfrak{B}_0$, $\lim_{t\to 0} A(t)=A$, and $\lim_{t\to 0} Fu(A(t))=B$. If this B is independent of the choice of $A(t)$, then $A\epsilon\mathfrak{B}$ is said to be a regular point of the matrix function. It is proved that if A is regular, then $Fu(A)$ commutes with every matrix C which commutes with A, and that $Fu(A)$ commutes with A even when A is not regular.

Three normal forms of a general matrix function are given. One of these takes the form $B=\sum_{k=0}^{n-1}c_kA^k$, where n is the order of A and in which the coefficients c_k are certain functions of the traces of $A, \cdots, A^n$. This normal form is available only when A has distinct latent roots, but the other two normal forms may be employed so long as the canonical form of A is diagonal. All three may fail if the canonical form is not diagonal. The difficulties inherent in the case of a nondiagonal canonical form can be overcome in the special case of an analytic matrix function $B=f(A)$, which is obtained by the formal replacement of the variable x by the matrix $A\epsilon\mathfrak{B}$ in a single valued function $f(x)$ which is analytic at the latent roots of A. It is proved that if A has latent roots λ_ν with multiplicities n_ν, $\nu=1, \cdots, m$, and if $f(x)$ is analytic at each λ_ν, then $A\epsilon\mathfrak{B}$ and

$$f(A)=\sum_{\nu=1}^{m} G_\nu \sum_{k=0}^{n_\nu-1} \frac{1}{k!} f^{(k)}(\lambda_\nu)(A-\lambda_\nu E)^k,$$

where $G_\nu=H_\nu(A)\prod_{\mu\neq\nu}(A-\lambda_\mu E)^{n_\mu}$, and the polynomials $H_\nu(x)$ are determined by $H_\nu(x)\prod_{\mu\neq\nu}(x-\lambda_\mu)^{n_\mu}\equiv 1 \pmod{(x-\lambda_\nu)^{n_\nu}}$, and E is the unit matrix. This theorem leads to a criterion for the reality of $f(A)$ when A is real.

In the case where $y=f(x)$ is the inverse function of a single valued analytic function $x=\phi(y)$, the condition on A for the existence of a solution B of $B=f(A)$ is given. A procedure is described for obtaining the most general B when A satisfies this condition. It is also proved that if $f(A)$ is an analytic matrix function and if $\delta f(A)=F(A+tD)-f(A)$, where t is a parameter and D is a matrix, then $\delta f(A)=tf'(A)D+O(t^2)$ if A and D commute, while for all D and for any matrix G which commutes with A, trace $\{G(\delta f(A)-tf'(A)D)\}=O(t^2)$.
D. E. Rutherford (St. Andrews).

Cherubino, Salvatore. Sulle matrici infinite. Ann. Scuola Norm. Super. Pisa (3) **3** (1949), 133–159 (1950).

The purpose of this paper is to provide a systematic exposition of the theory of matrices both finite and infinite, developing, where possible, both theories by parallel arguments. These arguments are naturally based, in the first instance, on the concept of linear dependence. The rows of a matrix A are defined to be linearly dependent if a nonvanishing row vector x can be found, such that $xA=0$. Since, in the case of infinite A, the x is not required to have only a finite number of nonzero elements, this definition is more embracing (more restricted independence) than the usual one and would allow the rows of the infinite matrix $a_{ij}=(j-1)^{i-1}$ to be regarded as dependent. Associativity of products is tacitly assumed but it is not clear from which premises this is derived.

Right and left inverses and norms are introduced, as are also one-sided unitary matrices. A section on symmetric and Hermitian (antisymmetric) matrices leads up to the diagonalisation of infinite Hermitian matrices under conjunctive transformations. Doubly- and multiply-infinite matrices are defined with a view to demonstrating the truth of $|AB|=|A||B|$ for infinite matrices, assuming of course that these determinants are convergent. Since the author (p. 153) adopts Hill's definition of an infinite determinant, his statement (p. 155), that a linear combination of an infinity of rows can be added to any row without altering the value of the determinant, is not true without further qualification. It is therefore not clear to the reviewer under what conditions the determinantal product theorem is valid. This product theorem is utilised in the final section in an extension, under certain conditions, to infinite matrices, of the Sylvester-Hadamard theorem relating the maximum modulus of a determinant to the norms of its rows (or columns). This has applications to the linear transformations of Hilbert space. *D. E. Rutherford* (St. Andrews).

Todd, J. A. The complete irreducible system of two quaternary quadratics. Proc. London Math. Soc. (2) **52**, 1–13 (1950).

Gordan [Math. Ann. **56**, 1–48 (1903)] gave a complete system of concomitants of two quaternary quadratics containing 580 terms. Most of these were reducible and Turnbull [same Proc. (2) **18**, 69–94 (1919)] attempted to obtain an irreducible complete system with a list of 125 forms. Of these, 3 were proved reducible by Williamson [J. London Math. Soc. **4**, 182–183 (1929)] and Turnbull subsequently reduced another 5 forms [ibid. **22**, 147–152 (1947); these Rev. **9**, 324]. In this paper Todd shows that 3 more of the forms are reducible and reaches finality by proving the other 114 forms to be irreducible.

The method used to prove irreducibility makes use of a method of S-functional analysis due to the reviewer. This shows that the totality of concomitants of degree p in the first, and degree q in the second quadratic corresponds to the terms in the expansion in terms of S-functions of $\{2\}\otimes\{p\}$, $\{2\}\otimes\{q\}$, involving the operation $\otimes$ called the plethysm of S-functions. The work is shortened by a consideration of concomitant types. The expansion gives reducible as well as irreducible concomitants, but allowance may be made for the former by considering products of irreducible concomitants of lower degree. The remaining terms must be irreducible. There is a possibility, however, that a syzygy may exist between reducible concomitants leaving room for an extra irreducible concomitant. The method gives a minimum system. The classical method gives a maximum system. When the systems coincide, as in this case, they must be both complete and irreducible. *D. E. Littlewood.*

Petr, K. On generating functions for the number of invariants belonging to one fundamental binary form. Rozpravy II. Třídy České Akad. **56**, no. 5, 23 pp. (1946). (Czech)

Cayley [Collected Mathematical Papers, v. 7, pp. 334–353 = Philos. Trans. Roy. Soc. London. Ser. A. **161**, 17–50

(1871)] introduced the numerical generating function, a rational expression in two variables a and t such that the coefficient of the term $a^\theta t^\mu$ in the development of the function gave the number of asyzygetic covariants of a binary $n$ic of the degree θ in the coefficients and order μ in the variables. Cayley and Sylvester subsequently developed the theory [e.g., Cayley, Collected Mathematical Papers, v. 10, pp. 339–400 = Philos. Trans. Roy. Soc. London. Ser. A. **169**, 603–661 (1878); Sylvester, Amer. J. Math. **2**, 223–251 (1879) = Collected Mathematical Papers, v. 3, pp. 283–311], particularly for the cases $n=5, 6, \cdots, 10$. The function is derived from

$$(t-t^{-1})/(1-at^n)(1-at^{n-2})(1-at^{n-4}) \cdots (1-at^{-n}).$$

Difficulties are met in its handling for the purpose of computing the actual number of irreducible covariants of indices θ, μ, owing to the negative terms that occur in the numerator; such terms indicate the presence of syzygies among such covariants of lower index. The present work is a refinement of this classical program wherein the numerical generating function is now resolved into certain partial fractions, for which the numerators, when $n=5, 6, 7, 8$, contain only positive terms. For $n=5, 6, 7, 8, 9, 10$, the number of such partial fractions is 2, 2, 3, 3, 4, 4, respectively. The method employed is skillful and systematic; the new expression is called the normalized generating function.

H. W. Turnbull (St. Andrews).

Petr, K. Generalization of results on the generating function giving the number of invariants for a binary form of given degree. Rozpravy II. Třídy České Akad. **56**, no. 8, 22 pp. (1946). (Czech)

This work continues the investigation of the normalized generating function (NGF) for a binary $n$ic [see the preceding review]. On writing $n=2m-1$ or $2m$, according to its parity, the author emphasizes the importance of m; the NGF has $m-1$ partial fractions in all cases. The cases $n=9, 10$ are calculated, and as before the terms in the numerators of the partial fractions are free from negative signs. It is conjectured that this happens for all values of n. In each partial fraction the denominator has n factors, each of the type $1-a^rt^s$. Each denominator is obtained by omitting $m-2$ factors from the least common multiple of the denominators which is a product of $n+m-2$ such factors. For example, with $n=7$, $(1-a^4)(1-a^6) \cdots (1-a^{12})(1-at)(1-at^3)(1-at^5)(1-at^7)$ has 9 factors. On removing from this product two consecutive factors of the consecutive set $(1-a^{10})(1-a^{12})(1-at)(1-at^3)$ in all possible ways (three in fact), the results are the required denominators. Rules for finding the numerators are given, but are much more elaborate.

H. W. Turnbull (St. Andrews).

Petr, K. On the generating function in normal form for the number of invariants of a binary form of degree 12. Rozpravy II. Třídy České Akad. **56**, no. 10, 16 pp. (1946). (Czech)

This continues the paper reviewed above by working out the NGF of a binary 12ic ($n=12$). Two different methods of calculation lead to the same results; again, the terms of the numerators are free from negative signs. Single generating functions, as originally developed by Cayley and Sylvester, have been computed, that for $n=12$ being the most advanced [see Sylvester, Amer. J. Math. **4**, 41–61 (1881) = Collected Mathematical Papers, v. 3, pp. 489–508; Franklin, Amer. J. Math. **3**, 128–153 (1880)].

H. W. Turnbull (St. Andrews).

Petr, K. On the generating functions for the number of invariants belonging to a system of two binary forms. Rozpravy II. Třídy České Akad. **56**, no. 14, 15 pp. (1946). (Czech)

This work continues the theory of normal generating functions (NGF) developed in the three papers reviewed above by extending the results to systems of two binary forms. If the generating function for one $n$ic form is written $(t-t^{-1})/f(a, n)$, that for an $m$ic and an $n$ic would be $(t-t^{-1})/f(a, m)f(b, n)$. This requires partial fractions to express its NGF, each of whose denominators consists of $m+n+1$ or less factors. In particular, the cases of two cubics, two quartics, and a cubic and a quartic are considered.

H. W. Turnbull (St. Andrews).

Jarušek, Jaroslav. A generating function for the number of invariants of a binary form of the eleventh order in Petr's normal form. Rozpravy II. Třídy České Akad. **57**, no. 8, 17 pp. (1947). (Czech)

The author calculates and tabulates the normalized generating function [cf. the fourth preceding review] of the binary form of order eleven by the methods of Petr. Five partial fractions A_i/B_i, $i=1, 2, 3, 4, 5$ are needed, where B_i is obtained by deleting four consecutive factors from the fifteen factors of $(1-a^4)(1-a^6) \cdots (1-a^{20})(1-at)(1-at^3) \cdots (1-at^{11})$, where B_1 has two, B_2 three, etc. factors involving t, of which $(1-at^9)(1-at^{11})$ are common to all the B_i. The tables for the numerators are highly elaborate.

H. W. Turnbull.

Foulkes, H. O. Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form. J. London Math. Soc. **25**, 205–209 (1950).

The author determines the types of the concomitants up to degree 4 in the coefficients of the ground forms, for the quintic and the sextic in any number of variables, by expanding the plethysms of S-functions $\{5\}\otimes\{2\}$, $\{5\}\otimes\{3\}$, $\{5\}\otimes\{4\}$, $\{6\}\otimes\{2\}$, $\{6\}\otimes\{3\}$, $\{6\}\otimes\{4\}$. A method of expansion has been given by the reviewer [Philos. Trans. Roy. Soc. London. Ser. A. **239**, 305–365 (1944); these Rev. **6**, 41]. It leads to quick calculation, but at various stages alternatives arise and it is not easy to make the correct choice. The author uses two methods to guide the choice in these difficult positions. The first is by the use of differential operators allied to S-functions [same J. **24**, 136–143 (1949); these Rev. **11**, 4]. The second makes use of an unproved hypothesis that for $n>m$ all terms in $\{n\}\otimes\{m\}$ appear in $\{m\}\otimes\{n\}$. Reasons are given which show that it is not possible that there should exist in any case a syzygy connecting the reducible concomitants accompanied by an extra irreducible concomitant of the same type.

D. E. Littlewood (Bangor).

Abstract Algebra

✱van der Waerden, B. L. Moderne Algebra. Teil I. 3d ed. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1950. viii+292 pp. 27 DM.

There have been two main changes. (1) The axiom of choice, present in the first edition but expunged in the second, has been reinstated. Transfinite induction is used rather than Zorn's lemma. (2) The chapter on valuations has been considerably expanded, and now contains a detailed discussion of valuations in algebraic function fields.

I. Kaplansky (Chicago, Ill.).

Lyndon, Roger C. The representation of relational algebras. Ann. of Math. (2) 51, 707–729 (1950).

A relation algebra (RA) is a Boolean algebra containing an identity I, a unary operation, and a binary operation with the following properties: (A1) $RI=R$; (A2) $R^{\cup\cup}=R$; (A3) $(RS)^{\cup}=S^{\cup}R^{\cup}$; (A4) $RS\cap T^{\cup}=0$ implies $ST\cap R^{\cup}=0$; (A5) $(RS)T=R(ST)$; (A6) $R\neq 0$ implies $V(RV)=V$. A proper RA (PRA) is an RA whose elements are relations and where the Boolean operations are the set-theoretical operations while the two additional operations are conversion and relative multiplication. An RA is said to be representable if it is isomorphic to a PRA. It is shown that there exists a finite RA which is not representable. An infinite set of conditions is given which are jointly necessary and sufficient in order for a finite RA to be representable. For infinite RA's which are complete and atomistic (regarded as Boolean algebras) these conditions are necessary but the question of their sufficiency is left open. Finally it is shown that there exists no set of algebraic conditions which characterize the class of all representable RA's.

B. Jónsson (Providence, R. I.).

Altwegg, Martin. Zur Axiomatik der teilweise geordneten Mengen. Comment. Math. Helv. 24, 149–155 (1950).

Six postulates for partial ordering are given in terms of the "betweenness" relation $(axb)\beta$, meaning $a\leqq x\leqq b$ or $a\geqq x\geqq b$; this solves problem 1 of the reviewer's "Lattice Theory" [Amer. Math. Soc. Colloquium Publ., v. 25, revised ed., New York, 1948; these Rev. 10, 673]. The last postulate involves an unlimited number of elements, but it is indicated that this defect is unavoidable. The new idea is to define $(ab)\gamma$ by $(aab)\beta$ but not $(aba)\beta$, so as to correspond to $a\leqq b$ or $a\geqq b$. One then notes, in effect, that if $(ab)\gamma$ and $(bc)\gamma$, the corresponding orderings are the same if $(abc)\beta$, and reversed otherwise. One then postulates that, in any cycle, the number of order-reversals is an even number.

G. Birkhoff (Cambridge, Mass.).

Benado, Mihail. The theory of metrizable structures. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 353–359 (1949). (Romanian. Russian and French summaries)

This note is concerned with lattices in which a function (x, y) is defined which is real-valued, nonnegative, zero if and only if $x=y$, and symmetric. The author imposes various subsidiary conditions, e.g., $(x, y)\leqq(x\cap y, x\cup y)$, and obtains under these conditions partial generalizations of theorems of Barbilian [Disquisit. Math. Phys. 5, 1–63 (1946); these Rev. 8, 432]. No proofs and few definitions are given.

E. Hewitt (Seattle, Wash.).

Fuchs, Ladislas. The meet-decomposition of elements in lattice-ordered semi-groups. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 105–111 (1950).

The author gives an abstract theory of meet-decompositions for ideals. He considers a commutative l-semigroup G which satisfies the ascending chain condition, and has a "meet-distributive" closure operation ϕ satisfying $\phi(x\cap y)=\phi(x)\cap\phi(y)$. [For terminology, see the reviewer's Lattice Theory, Amer. Math. Soc. Colloquium Publ., v. 25, revised ed., New York, 1948; these Rev. 10, 673.] An element $p\epsilon G$ is called ϕ-prime if $a_1\cdots a_k\leqq p$ implies that some $a_i\leqq\phi(p)$; it is called ϕ-primary if $a_1\cdots a_k\leqq p$ implies $a_1\leqq p$ or some $a_j\leqq\phi(p)$, $j=2,\cdots,k$; this makes every ϕ-primary element a ϕ-prime. It is shown that any element can be written as a meet of ϕ-prime components, the closures of the components being uniquely determined; in any shortest representation as a meet of ϕ-primary components, the components themselves are unique. Various special cases of interest are noted, including the case that $\phi(a)$ is the join of all x satisfying $x^n\leqq a$ for some $n=n(x)$.

G. Birkhoff (Cambridge, Mass.).

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Szele, T. Die Ringe ohne Linksideale. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 785–789 (1949).

[The author's name is misprinted as Szelle in the journal.] The author proves that a ring without proper left ideals is either a sfield or a nil-ring with p elements (p a prime). This result is not new.

S. A. Jennings.

Nakayama, Tadasi. Galois theory for general rings with minimum condition. J. Math. Soc. Japan 1, 203–216 (1949).

This paper gives the generalization to the case of general rings with minimum condition of noncommutative Galois theory for finite groups of outer automorphisms, which was begun by Jacobson [Ann. of Math. (2) 41, 1–7 (1940); these Rev. 1, 198] for the case of division rings and generalized to simple and uniserial rings by Azumaya [same vol., 130–146 (1949); these Rev. 11, 414].

Let R be a ring with an identity element which satisfies the minimum condition for left and for right ideals. If σ is an automorphism of R, let (R, σ) denote the two-sided R-module whose additive group is that of R and where the endomorphisms are defined by $r\rightarrow r*r_1=rr_1$ and $r\rightarrow r_1*r=\sigma(r_1)r$. Then a finite group G of automorphisms of R is called a Galois group if, for any two distinct elements $\sigma, \tau\epsilon G$, no nonzero residue class module of a submodule of (R, σ) is isomorphic with a residue class module of a submodule of (R, τ). Let S denote the subring of R consisting of the elements left fixed by the automorphisms of a Galois group G. The author obtains, among others, the following results: (1) R has a normal right (or left) basis over S; (2) if R is two-sided directly indecomposable, then G coincides with the group of all automorphisms of R which leave the elements of S fixed; (3) if R is two-sided directly indecomposable, then the set of subgroups of G is in the usual Galois correspondence with the set of those rings T between R and S over which R is right (and left) regular, in the sense that the direct sum of a certain number of copies of the T-module R is isomorphic with the direct sum of a certain number of copies of T. The methods of the author involve the use of generalized crossed products, the application of the Krull-Remak-Schmidt theorem to regular modules (giving, in particular, a very natural proof of the normal basis theorem), and the technique of translating the problems concerning automorphisms into questions concerning the corresponding rings of endomorphisms and their commutator rings.

G. Hochschild (Urbana, Ill.).

Cassels, J. W. S., and Wall, G. E. The normal basis theorem. J. London Math. Soc. 25, 259–264 (1950).

The authors prove the well-known theorem that if E is a normal (separable) extension of finite degree n of a field F then E contains an element β such that the n conjugates of β are linearly independent over F. The case in which F is infinite and that in which E is cyclic over F are treated separately.

E. R. Kolchin (New York, N. Y.).

Kaplansky, Irving. Forms in infinite-dimensional spaces. Anais Acad. Brasil. Ci. **22**, 1–17 (1950).

Soit D un corps (commutatif ou non), $\alpha \to \alpha^*$ un anti-automorphisme involutif de D sur lui-même, E un espace vectoriel à gauche sur D, (x, y) une fonction définie dans $E \times E$, à valeurs dans D, additive par rapport à chacune des variables et telle que $(\alpha x, y) = \alpha(x, y)$, $(x, \alpha y) = (x, y)\alpha^*$; en outre on suppose que $(y, x) = (x, y)^*$ ou $(y, x) = -(x, y)^*$; la relation d'orthogonalité $(x, y) = 0$ est alors symétrique. L'auteur considère uniquement le cas où E admet une base dénombrable sur D; lorsque la forme (x, y) est alternée (c'est-à-dire que $(x, x) = 0$ identiquement), il montre qu'il existe dans E une base symplectique pour cette forme. Lorsque D est commutatif et de caractéristique $\neq 2$, $\alpha^* = \alpha$ et $(y, x) = (x, y)$ (forme symétrique), il y a encore une base orthonormale pour la forme lorsque D est tel que pour un entier fixe k, toute forme quadratique non singulière à k variables représente 1 dans D (c'est ce qui se passe lorsque D est fini pour $k = 2$, ou lorsque D est un corps p-adique, avec $k = 4$); on a un résultat analogue lorsque toute forme quadratique non singulière à k variables représente 1 ou -1 (cas du corps des nombres rationnels pour $k = 4$). L'auteur donne ensuite une démonstration simple du théorème de Witt sur l'extension à E d'un isomorphisme (pour la structure définie par (x, y)) entre deux sous-espaces V_1, V_2 de dimension finie, et pose le problème de la généralisation de ce résultat lorsque V_1 et V_2 sont de dimension infinie; il obtient deux résultats partiels dans cette direction pour les formes alternées. *J. Dieudonné* (Baltimore, Md.).

Chabauty, Claude. Sur la théorie des fonctions dans un corps valué. I. C. R. Acad. Sci. Paris **231**, 396–397 (1950).

This paper deals with fields F complete with a valuation. The following two definitions are made: F is called formally complex (f.c.) if

$$\sup_{|x_{ij}| \leq 1} (|\det (x_{ij})|) = \lim_{\epsilon \to 0} (\sup_{1-\epsilon \leq |x_i|, |x_j| \leq 1} \prod |x_i - x_j|),$$

$i, j = 1, \cdots, n$, and formally real if the continuous functions on $|x| \leqq 1$ are uniform limits of polynomials. The former are characterized as either the complex numbers or non-Archimedean nonlocally compact fields; the latter as the remaining ones [see Dieudonné, Bull. Sci. Math. (2) **68**, 79–95 (1944); these Rev. **7**, 111]. Another result is that Cauchy's inequalities for coefficients in Taylor's series are valid for all f.c. fields [see Schöbe, Beiträge zur Funktionentheorie in nichtarchimedisch bewerteten Körpern, Universitas-Archiv 42, Helios Verlag, Münster, 1930] and that, with suitable hypotheses about the radii of convergence, uniform limits of functions representable by Taylor's series are themselves so representable [see Schnirelman, Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **1938**, 487–498; Loonstra, Thesis, University of Amsterdam, 1941; Krasner, same C. R. **222**, 37–40, 165–167, 363–365, 581–583 (1946); these Rev. **7**, 111, 429]. The possibility of several generalizations is mentioned (several variables, Laurent's series, variables and coefficients in Banach spaces or algebras over F). *G. K. Kalisch.*

Chabauty, Claude. Sur la théorie des fonctions dans un corps valué. II. C. R. Acad. Sci. Paris **231**, 432–434 (1950).

This is a continuation of a previous note [see the preceding review for definitions and references] whose results form the basis for the following discussion: Liouville's theorem is true for f.c. fields [see Schöbe, loc. cit.]; analytic continuation is (nontrivially) defined for non-Archimedean algebraically closed fields [this includes Krasner's results, loc. cit.]; integration of analytic functions in a f.c. is defined and turns out to be the Cauchy integral in the case of the complex numbers, and the Schnirelman integral otherwise [see Schnirelman, loc. cit., and Loonstra, loc. cit.]. *G. K. Kalisch* (Minneapolis, Minn.).

★Steinitz, Ernst. Algebraische Theorie der Körper. Chelsea Publishing Co., New York, N. Y., 1950. 176 pp. (1 plate).

A photographic reprint of the edition of Steinitz' paper [J. Reine Angew. Math. **137**, 167–309 (1910)], together with an appended Abriss der Galoisschen Theorie, prepared by R. Baer and H. Hasse and published in 1930 by Walter de Gruyter, Berlin-Leipzig.

Goldie, A. W. The Jordan-Hölder theorem for general abstract algebras. Proc. London Math. Soc. (2) **52**, 107–131 (1950).

The Jordan-Hölder and Schreier-Zassenhaus theorems are partially generalized to "weakly permutable" congruence relations on an arbitrary abstract algebra A with least subalgebra E. The generalizations are somewhat involved, but may be roughly described as follows. Two equivalence relations R and S are called "weakly permutable" when xRy and ySe for some $y \epsilon A$ and $e \epsilon E$ if and only if xSy_1 and y_1Re_1 for some $y_1 \epsilon A$ and $e_1 \epsilon E$. The "kernel" $\{R\}$ of a congruence relation R is defined as the set of R-classes whose intersection with E is nonvoid. The generalizations are to the "quotient-algebras" $\{R_i\}/R_{i+1}$ of chains of congruence relations R_i. They include weakened generalizations of the results on pp. 87–89 of the reviewer's "Lattice Theory" [Amer. Math. Soc. Colloquium Publ., v. 25, revised ed., New York, 1948; these Rev. **10**, 673], under hypotheses weak enough to apply to the systems previously considered by Hausmann and Ore [Amer. J. Math. **59**, 983–1004 (1937)] and Murdoch [ibid. **61**, 509–522 (1939)]. *G. Birkhoff* (Cambridge, Mass.).

Žukov, A. I. Nonassociative free decompositions of algebras with a finite number of generators. Mat. Sbornik N.S. **26(68)**, 471–478 (1950). (Russian)

Let S be a free (nonassociative) algebra with a finite number of generators, and suppose it has a homomorphic image which is a free product of algebras $A_1, \cdots, A_k$. The author's main theorem asserts that it is possible to choose free generators for S, each of which maps into one of the A_i's. This theorem was suggested by an analogous one for groups proved by Gruschko [Rec. Math. [Mat. Sbornik] N.S. **8(50)**, 169–182 (1940); these Rev. **2**, 215]. There are several corollaries of which the following two are typical: (1) Every finitely generated algebra is a free product of a finite number of indecomposable algebras; (2) in a free algebra with n generators, every system of n generators is free. This work was inspired by, and uses the methods of Kurosh [ibid. **20(62)**, 239–262 (1947); these Rev. **9**, 5]. *I. Kaplansky* (Chicago, Ill.).

Dynkin, E. B. Regular semisimple subalgebras of semisimple Lie algebras. Doklady Akad. Nauk SSSR (N.S.) **73**, 877–880 (1950). (Russian)

A "regular" subalgebra of a semi-simple Lie algebra G (over an algebraically closed field of characteristic 0) is one having a basis consisting of elements of a Cartan subalgebra of G, together with corresponding root vectors. Any sub-

algebra containing a regular element is itself regular, but the converse is false. The author states and includes sketches of proof for a number of results concerning regular subalgebras. In particular, any regular subalgebra can be extended to a subalgebra containing a regular element, and different from G. The author points out that this implies a result due to Borel and de Siebenthal [Comment. Math. Helv. **23**, 200–221 (1949); these Rev. **11**, 326]. His chief tool is the concept of P-system, this being a set of roots of G which can be regarded (in a fashion concerning which the author is not explicit) as the system of all prime roots of a subalgebra of G. A set Γ of roots is stated to be a P-system if and only if (1) $\alpha, \beta \epsilon \Gamma$ implies $\alpha-\beta$ is not a root, (2) Γ is linearly independent. If G has rank n, then every P-system is contained in a P-system consisting of n elements, and every system of the latter type can be obtained from the system of prime roots of G by a chain of elementary transformations. *I. E. Segal* (Chicago, Ill.).

Malcev, A. I. Solvable Lie algebras. Amer. Math. Soc. Translation no. **27**, 36 pp. (1950).

Translated from Bull. Acad. Sci. URSS. Sér. Mat. [Izvestiya Akad. Nauk SSSR] **9**, 329–356 (1945); these Rev. **9**, 173.

Okugawa, Kôtaro. Basis theorem for D-polynomials. Math. Japonicae **2**, 35–39 (1950).

The basis theorem for differential polynomials proved first by Ritt in the analytic case [see Differential Equations from the Algebraic Standpoint, Amer. Math. Soc. Colloq. Publ., v. 14, New York, 1932], then by Raudenbush in the abstract case [Trans. Amer. Math. Soc. **36**, 361–368 (1934); see also Ritt, Differential Algebra, Amer. Math. Soc. Colloq. Publ., v. 33, New York, 1950; these Rev. **12**, 7], and in a more general setting by the reviewer [Trans. Amer. Math. Soc. **52**, 115–127 (1942); these Rev. **3**, 264] is proved anew. In the present formulation the differential ring of coefficients is assumed to contain a field of characteristic 0, the unity element of which is a unity for the ring.
E. R. Kolchin (New York, N. Y.).

Theory of Groups

Turing, A. M. The word problem in semi-groups with cancellation. Ann. of Math. (2) **52**, 491–505 (1950).

In this paper a substantial further step is taken towards determining the status of the word problem for groups. A "semigroup with cancellation," with a finite number of generators and relations, is constructed for which the word problem is unsolvable; that is, a system with an associative multiplication satisfying the laws: $AB=AC$ implies $B=C$, and $BA=CA$ implies $B=C$. The unsolvability has previously been proved by Post [J. Symbolic Logic **12**, 1–11 (1947); these Rev. **8**, 558] for associative systems without cancellation. As in Post's paper, the word problem is referred back to the original unsolvable problem on a Turing machine, the relations being so chosen that any equivalence between words obtainable with the help of cancellations is also obtainable from the relations without cancellation. There are a large number of misprints. In the table at the foot of p. 495 all commas except the first in the "first phase relations," and all except the last in the "second phase relations," should be primes applied to the preceding suffix; e.g., the first of the "first phase relations" should read $(v_i k_h, \sigma_m j_h{}' v_{i'} \tau_m)$. At l. 4, p. 499, $(\phi_r(C), H_r)$ should be $(\phi_1(C), H_R)$. *M. H. A. Newman* (Manchester).

Russo, Salvatore. Legge di moltiplicazione delle sostituzioni di un gruppo G_s, con la S prodotto di due cicli dello stesso ordine. Matematiche, Catania **4**, 61–63 (1949).

The multiplication table of the centralizer in the symmetric group of degree $2r$ of the permutation

$$(a_1 a_2 \cdots a_r)(a_{r+1} a_{r+2} \cdots a_{2r})$$

is constructed. *S. A. Jennings* (Vancouver, B. C.).

Szép, J. On the structure of groups which can be represented as the product of two subgroups. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 57–61 (1950).

Let $\mathfrak{G}$ be a finite group with subgroups $\mathfrak{H}$, $\mathfrak{K}$ such that every element of $\mathfrak{G}$ may be written in the form HK, $H \epsilon \mathfrak{H}$, $K \epsilon \mathfrak{K}$. In an earlier paper [Comment. Math. Helv. **22** (1949), 31–33 (1948); these Rev. **10**, 181] the author investigated groups of this type, and the present paper is a continuation of this study. Let $\Pi(\mathfrak{K})$ be the group of the permutations π_H of the elements of $\mathfrak{K}$, defined in the paper cited above, and similarly for $\Pi(\mathfrak{H})$. If $\Pi(\mathfrak{K})$ is an automorphism group of $\mathfrak{K}$, and if $\Pi(\mathfrak{H})$ is an automorphism group of $\mathfrak{H}$, then $\mathfrak{G}$ is said to be "of automorphic composition," while if only one of these statements is true, $\mathfrak{G}$ will be said to be "of semi-automorphic composition." Among the results obtained are the following. (i) Every $\mathfrak{G}$ of automorphic or semi-automorphic composition has a proper normal subgroup; (ii) if $\mathfrak{G}$ is of automorphic composition, and if $\mathfrak{M}$, $\mathfrak{N}$ are the maximal normal subgroups of $\mathfrak{G}$ contained in $\mathfrak{H}$, $\mathfrak{K}$ respectively, then $\mathfrak{G}/(\mathfrak{M}\times\mathfrak{N}) \cong (\mathfrak{K}/\mathfrak{N})\times(\mathfrak{H}/\mathfrak{M}) \cong \Pi(\mathfrak{H})\times\Pi(\mathfrak{K})$; (iii) if the orders of $\mathfrak{H}$, $\mathfrak{K}$ are relatively prime, every normal subgroup of $\mathfrak{G}$ is a subgroup of $\mathfrak{H}$, or of $\mathfrak{K}$, or is of the form $\bar{\mathfrak{H}}\bar{\mathfrak{K}}$, where $\bar{\mathfrak{H}}$, $\bar{\mathfrak{K}}$ are normal subgroups of $\mathfrak{H}$, $\mathfrak{K}$ respectively; (iv) if the orders of $\mathfrak{H}$, $\mathfrak{K}$ are relatively prime, and if $\mathfrak{H}$, $\mathfrak{K}$ are simple with $\Pi(\mathfrak{K})\cong\mathfrak{H}$, $\Pi(\mathfrak{H})\cong\mathfrak{K}$, then $\mathfrak{G}$ is simple.
S. A. Jennings (Vancouver, B. C.).

Szép, J. On simple groups. Publ. Math. Debrecen **1**, 98 (1949).

The author remarks that if a simple group G has a subgroup H of prime index p then G is the product of H and a Sylow subgroup of order p. *S. A. Jennings.*

Herstein, I. N. On a conjecture on simple groups. Proc. Amer. Math. Soc. **1**, 438–439 (1950).

Let $\mathfrak{N}_p$ be the radical of the group ring of the finite group $\mathfrak{G}$ (of order g) over a field of characteristic p. Then $\mathfrak{N}_p \neq 0$ if and only if $p|g$. The author proves that the famous conjecture that every $\mathfrak{G}$ of odd order is solvable is equivalent to the following: For some prime $p|g$ there exists at least one element $G \epsilon \mathfrak{G}$, $G \neq 1$, such that $G \equiv 1 \bmod \mathfrak{N}_p$ in the group ring of $\mathfrak{G}$ over any field of characteristic p.
S. A. Jennings (Vancouver, B. C.).

Dyubyuk, P. E. On the number of subgroups of given index of a finite p-group. Mat. Sbornik N.S. **27**(69), 129–138 (1950). (Russian)

The author continues his study of the number of subgroups of given index in a finite p-group. [For the Abelian case see the author, Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 351–378 (1948); these Rev. **10**, 98; for the general background see Zassenhaus, Lehrbuch der Gruppentheorie, v. 1, Teubner, Leipzig-Berlin, 1937, chapter IV, § 4.] In

what follows let $\mathfrak{P}$ be a p-group of order p^n, and let $\mathfrak{A}$ be a subgroup of $\mathfrak{P}$ lying between the first derivative $\mathfrak{K}$ and the Frattini subgroup Φ. The order of $\mathfrak{P}/\mathfrak{A}$ is to be p^l and the orders of its elements are to have least upper bound p^h. Typical of the results are the following. If i and α are positive integers so chosen that $0 \leqq \alpha \leqq l-(i-1)h$, then the number of subgroups of $\mathfrak{P}$ of index p^α is congruent to $\varphi_{l,\alpha}$ modulo p^i (where $\varphi_{l,\alpha}$ is defined as in Zassenhaus). The modulus p^i may be replaced by p^j, where $j=[(l-\alpha)/n]+1$ and $0 \leqq \alpha$. This specializes to a result due to P. Hall [Proc. London Math. Soc. (2) 36, 29–95 (1933)] by taking $\mathfrak{A}=\Phi$. If $\mathfrak{P}$ is not cyclic and if $\mathfrak{A}=\mathfrak{K}$, $0<\alpha<l$, then the number of subgroups of $\mathfrak{P}$ of index p^α is congruent to $(1+p)$ modulo p^2. This latter generalizes results due to Kulakoff [Math. Ann. **104**, 778–793 (1931)] and Miller [Bull. Amer. Math. Soc. **26**, 66–72 (1919)]. *F. Haimo* (St. Louis, Mo.).

Grün, Otto. Beiträge zur Gruppentheorie. IV. Über eine charakteristische Untergruppe. Math. Nachr. **3**, 77–94 (1949).

[Parts I, II, III have appeared earlier [J. Reine Angew. Math. **174**, 1–14 (1935); **186**, 165–169 (1945); same Nachr. **1**, 1–24 (1948); these Rev. **10**, 504] but the present part is quite independent of those preceding.] For a given group A and positive integer m let $H_m=H_m(A)$ be the subgroup of A generated by elements of the form $(st)^{-m}s^mt^m$, where $s, t \epsilon A$. Then H_m is a characteristic subgroup, the smallest normal subgroup of A such that its factor group has the property that the mth power of a product is the product of the mth powers. Moreover, $H_1=A$ and $H_2=(A, A)$, while if $x, y \epsilon A$ then $(x^m, y^{m-1}) \epsilon H_m$. The author investigates the group H_m when A is a free group F, and also when A is of the form F/N, particularly a finite group. In what follows let F be a free group, G a finite group, $F(q)$, $G(q)$ the groups generated by the qth powers of elements of F, G respectively, $F_2=(F, F)$, $G_2=(G, G)$, and $H_m=H_m(F)$, $H_m^*=H_m(G)$. The discussion for free groups is simplest, since then $F(q)$, F_2, and H_m are proper subgroups of F. Using only elementary commutator algebra, the author obtains many relations connecting F_2, $F(q)$, and H_m. For example, the subgroup H_m is identified in the form $H_m=F_2 \cap F(m-1) \cap F_m$, while $F_2=H_{m-1}H_mH_{m+1}$. It is shown that any element of F_2/H_m is of finite order dividing $m(m-1)$, $F_2/H_m \cong (F_2 \cap F(m-1)/H_m) \times (F_2 \cap F(m)/H_m)$, while the order of any element of $F_2/(H_{m-1} \cap H_m \cap H_{m+1})$ divides $(m-2)(m-1)m(m+1)$. Moreover, $F(m) \cap F(m-1)/H_m$ lies in the center of F/H_m, this center being of infinite order. The kernel of the homomorphism $xH_m \to x^mH_m$ of F/H_m upon $F(m)/H_m$ is $F_2 \cap F(m-1)$, while if $\bar{F}_2(m)$ is the group generated by the mth powers of elements of F_2, then the commutator group of $F/F(m)$ is isomorphic to a certain factor group of $F_2/\bar{F}_2(m)$. Of interest also is the remark that if $s_1, \cdots, s_k$ generate F, then $s_1^m, \cdots, s_k^m$ together with certain elements of H_m generate the free group $F(m)$.

In the second part of the paper the author shows that most of the above relations still hold when F is replaced by F/N, although some of them become trivial. However, if F is replaced by a finite group G then more precise relations may be obtained in certain cases. We mention only one example. Let n_1 be the exponent of G_2 (the least common multiple of the orders of the elements of G_2). Then every element of the factor group $G_2/(H_{m-1} \cap H_m \cap H_{m+1})$ has an order which divides $(\frac{1}{2}(m-1)m(m+1), n_1)$ (cf. the corresponding result for F mentioned above). *S. A. Jennings* (Vancouver, B. C.).

Skopin, A. I. The factor groups of an upper central series of free groups. Doklady Akad. Nauk SSSR (N.S.) **74**, 425–428 (1950). (Russian)

Let G be the free group with k free generators. Then let $G=G_0 \supset G_1 \supset \cdots \supset G_n \supset \cdots$ be an upper central series such that: (1) G_{i-1}/G_i is an elementary Abelian p-group, $p>2$; (2) G_{i-1}/G_i is in the center of G/G_i; and (3) G_i is minimal with properties (1) and (2). The author investigates the factor groups G_{n-1}/G_n. He uses the Magnus representation of G in terms of formal power series generated by $1+x_i$, $(1+x_i)^{-1}=1-x_i+x_i^2-\cdots$, with $i=1, \cdots, k$. He shows that the elements of G of the form $1+\sum_{i=1}^{n+1}p^{n-i+1}P_i+\cdots$ are precisely the elements of G_n. From this it follows that G_{n-1}/G_n is isomorphic to the additive group modulo p of all Lie polynomials in $x_1, \cdots, x_k$ of degrees 1 through n inclusive. *M. Hall* (Washington, D. C.).

Kaloujnine, Léo. Sur les sous-groupes centraux d'un produit complet de groupes abéliens. C. R. Acad. Sci. Paris 229, 1289–1291 (1949).

Kaloujnine, Léo. Caractérisation de certains sous-groupes centraux d'un produit complete de groupes abéliens. C. R. Acad. Sci. Paris **230**, 1633–1634 (1950).

In an earlier note [same C. R. **227**, 806–808 (1948); these Rev. **10**, 351] Krasner and the author defined the "complete product" $\Gamma=\Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_s$ of abstract groups $\Gamma_1, \cdots, \Gamma_s$ by identifying them with their regular representations and forming the complete product of these permutation groups [cf. the review cited above for definitions and notations]. Let $e_1, \cdots, e_s$ be the identity elements of $\Gamma_1, \cdots, \Gamma_s$ (considered as permutation groups), let G be a transitive subgroup of Γ and let G_i be the subgroup of G such that $\sigma(e_1, \cdots, e_i, e_{i+1}, \cdots, e_s)=(e_1, \cdots, e_i, *, \cdots, *)$ for all $\sigma \epsilon G_i$. For $i=1, \cdots, s$ we obtain a series $G=G_0 \supset G_1 \supset \cdots \supset G_s$, the "canonical series" of G. In the present notes the author is concerned with the problem of when the canonical series of a transitive subgroup G of Γ is a central series. For this to be true it is shown that the $\Gamma_1, \cdots, \Gamma_s$ must be Abelian. If the Γ_i are Abelian, the following theorem is proved. A necessary and sufficient condition that the canonical series of a transitive subgroup G of the complete product $\Gamma=\Gamma_1 \circ \cdots \circ \Gamma_s$ be a central series is that the centralizer of G in Γ be also transitive. In the second note a necessary and sufficient condition for the canonical series to be actually the upper central series of G is obtained. The condition requires that certain homomorphisms of the G_i be isomorphisms, but for specific details reference should be made to the original note. *S. A. Jennings* (Vancouver, B. C.).

Kaloujnine, Léo. Sur quelques propriétés des groupes d'automorphismes d'un groupe abstrait. (Généralisation d'un theorème de M. Ph. Hall). C. R. Acad. Sci. Paris **231**, 400–402 (1950).

Let $G=G_0 \supseteq G_1 \supseteq \cdots \supseteq G_s \supseteq G_{s+1}=1$ be a chain of normal subgroups of G. It is shown that a group A of automorphisms of G leaving every factor group G_i/G_{i+1} fixed is nilpotent of class s or less. The A-commutator group of a subgroup H, written $C(H; A)$ is the subgroup generated by all elements $h^\alpha h^{-1}$, $\alpha \epsilon A$. Now consider a chain $G=C^0(G; A) \supseteq \cdots \supseteq C^r(G; A)=1$, where $C^i(G; A)$ is the ith iterated A-commutator group. If each of these is a normal subgroup of G, then $C^1=C^1(G; A)$ is nilpotent and $C^1 \supseteq C^2 \supseteq \cdots \supseteq 1$ is a central series for C^1. This generalizes a theorem of P. Hall, which may be regarded as the special case in which A is the group of inner automorphisms of G. *M. Hall* (Washington, D. C.).

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Szele, T. Die unendliche Quaternionengruppe. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 799–802 (1949).

[The author's name is misprinted as Szelle in the journal.] Let Q_n ($n\geqq 3$) be the "generalized quaternion group" of order 2^n, defined by the relations $A^{2^{n-1}}=1$, $BAB^{-1}=A^{-1}$, $B^2=A^{2^{n-2}}$, and let Q_∞ be the group generated by the elements $C, D_1, D_2, \cdots$ with the relations $D_1{}^2=1$, $C^2=D_1$; $CD_kC^{-1}=D_k{}^{-1}$, $D^2{}_{k+1}=D_k$, $k=1, 2, \cdots$. The author proves that Q_∞ is uniquely defined as the smallest group such that it contains subgroups isomorphic to Q_n for all $n\geqq 3$, each $Q_n\subset Q_\infty$ being itself a subgroup of a $Q_{n+1}\subset Q_\infty$.

S. A. Jennings (Vancouver, B. C.).

Todd, J. A. The characters of a collineation group in five dimensions. Proc. Roy. Soc. London. Ser. A. **200**, 320–336 (1 plate) (1950).

The subject of this paper is the collineation group G of order 6531840 which has recently been investigated by several authors [C. M. Hamill, unpublished manuscript; the author, Proc. Cambridge Philos. Soc. **46**, 73–90 (1950); E. M. Hartley, ibid., 91–105 (1950); these Rev. **11**, 578]. The present paper utilizes Hamill's results, already summarized by the author in his cited paper, and Hartley's investigations arise directly from it. The author begins by describing a new tentative method for computing the characters of any group G; and more especially, in the first place, the rationally irreducible characters $\theta^{(i)}$ of G, in number r_1 say. Associating any subgroup H of G with the rational character of the representation of G as a permutation group of the cosets of H in G, the author makes his method depend on the selection of a set of r_1 known subgroups of G, such that the r_1 associated characters admit of representation as independent linear forms in the $\theta^{(i)}$ with nonnegative integral coefficients. For the determination of these coefficients and hence of the $\theta^{(i)}$ in terms of the characters of the known subgroups, the author obtains a set of Diophantine equations which may have a unique solution, or only one solution which cannot be rejected on obvious grounds. The method, though admittedly tentative, is interesting because of its generality, but it requires a knowledge of subgroups of G which should preferably be of higher order. It is applied with complete success to the particular group G which is the subject of this paper. The detailed analysis of this group suggests also the appropriate technique for solving the further problem of separating the characters $\theta^{(i)}$ into their absolutely irreducible components.

The rest of the paper is concerned with the particular group G, which is generated by 126 projections, contains 34 classes of conjugate operations, and has 31 sets of conjugate cyclic subgroups; and also with its subgroup G', generated by products of an even number of projections, which is simple and separates the classes of operations of G into even and odd classes. The main results are given in tables. Table I gives the distributions, among the conjugate sets, of the operations of G and 16 of its subgroups, and of G' and 16 of its subgroups. Table II gives the 33 corresponding rational characters of which 31 are linearly independent. Table IV, to which table III is auxiliary, lists the 31 rationally irreducible characters and the 34 irreducible characters of G; and table V lists the sets of analogous characters, in number 20 and 17, of G'.

The last section of the paper is concerned mainly with the use of the tables, by Littlewood's method [The Theory of Group Characters and Matrix Representations of Groups, Oxford University Press, 1940, chapter IX; these Rev. **2**, 3], to discover subgroups of G and G'. In particular, the author finds all the subgroups of G' whose index does not exceed 200, and the groups of G in which they may be contained as subgroups of index 2. It appears that of the four largest subgroups of G', only two were readily deducible from available descriptions of the configuration of the 126 vertices, a third had previously been obtained by difficult investigation, while the existence of the fourth, since confirmed by Hartley, had not been contemplated until its existence was indicated by the table of characters.

J. G. Semple (London).

Suškevič, A. K. On the construction of some types of groups of infinite matrices. Zapiski Naučno-Issled. Inst. Mat. Meh. Har'kov. Mat. Obšč. (4) **19**, 27–33 (1948). (Russian)

A left semiorthogonal matrix A is an infinite bounded matrix whose row vectors are an incomplete set of orthogonal vectors in Hilbert space. If A' is the transpose of A, then $AA'=E$ and $A'A=E_1\neq E\cdot E_1$ is idempotent. The equations $YA'=E$ obviously have the solution $Y=A$ and if $a_\alpha{}^{(\lambda)}$ is any set of mutually orthogonal vectors orthogonal to the row vectors a_α of A, then $Y_\alpha{}^{(\lambda)}=a_\alpha+a_\alpha{}^{(\lambda)}$ is also a solution and a_α, $Y_\alpha{}^{(\lambda)}$ are linearly independent. The next step is the construction of the solutions of the system $AX=E$; $Y^{(\lambda)}X=P_{(\lambda)}$, where $P_{(\lambda)}$ is a set of arbitrary bounded matrices. This enables the author to build from any left proper semiorthogonal matrix K_1 a sequence of r matrices $K_1, \cdots, K_r$ and another sequence of s matrices $L_1=K_1'$, $L_2, \cdots, L_s$, such that $K_1L=KL_1=E$ and $K_\alpha L_\lambda=P_{\alpha\lambda}$, where $P_{\alpha\lambda}$ are arbitrary orthogonal matrices. Then starting with an arbitrary generalized group G the author builds a number of element-complexes of the form $L_\lambda GK_1$ and L_1GK_α and shows that these complexes are generalized groups simply isomorphic to G.

M. S. Knebelman (Pullman, Wash.).

Ritt, J. F. Differential groups and formal Lie theory for an infinite number of parameters. Ann. of Math. (2) **52**, 708–726 (1950).

By a differential group of order n is meant an associative differential operation (of rank n) $z_i=\mathfrak{A}_i(u_1, \cdots, u_n; v_1, \cdots, v_n)$, $i=1, \cdots, n$ [see the author, same Ann. (2) **51**, 756–765 (1950); these Rev. **11**, 639]. This paper first gives the theory of formal Lie groups of infinitely many parameters; the theory for a finite number of parameters has been given by Bochner [ibid. **47**, 192–201 (1946); these Rev. **7**, 413]. Let a formal Lie group of infinitely many parameters be given by $z^i=f^i(u, v)$ $(i=1, 2, \cdots)$, where the power series f^i starts with u^i+v^i and each higher term involves u and v; $u^i=f^i(u, 0)$, $v^i=f^i(0, v)$, $f^i(f(u, v), w)=f^i(u, f(v, w))$. The structure constants are introduced by $c^i_{jk}=a^i_{jk}-a^i_{kj}$, where a^i_{jk} is the coefficient of u^jv^k in f^i, and satisfy the usual anticommutativity and Jacobi relations. For each i almost all c^i_{jk} are 0. The proof of the converse theorem starts with solving $dw_j{}^i(t)/dt=\delta_j{}^i+c^i_{pq}u^pw_j{}^q(t)$ as usual, but formally. Then, $w_j{}^i(t)$ with $t=1$ gives a solution $\alpha_j{}^i(u)$ of the Cartan-Maurer equation. To solve $\beta_p{}^i\alpha_j{}^p=\alpha_p{}^i\beta_j{}^p=\delta_j{}^i$ a norm is introduced as the inverse of the lowest degree in a series. The system $\partial f^i/\partial u^j=\beta_p{}^i(f)\alpha_j{}^p(u)$ is solved by a successive process dealing with a finite number of u^i at each step. The function f thus obtained gives a desired group with given c^i_{jk}. The uniqueness holds in the sense of a substitution $z^i=z_1{}^i+V^i(z_1)$ (V with terms of 2nd and higher degrees); similarly for u, v.

With a differential group $z_i=\mathfrak{A}_i(u, v)$, let z^{ij} be the jth derivative of z_i, and similarly with u, v. Thus a (nondifferential) group $z^{ij}=\mathfrak{A}^{ij}(u, v)$, with double indices, is ob-

tained. Let $c_{j\alpha\beta}^{ip}$ be its structure constants. The c's with $p=0$ are called the structure constants of the differential group. Other c's are expressed by those and their derivatives. The Jacobi relations give rise to certain quadratic relations for the structure constants c^{i0} and their derivatives. For the converse theorem for differential groups the author constructs a differential group with a given system of structure constants (satisfying the above-mentioned quadratic relations together with the anti-commutativity and vanishing except in finite number). To prove it, the system is amplified by c^{ip} with $p\neq 0$ so that the associativity is valid through terms of 3rd degree, securing thereby the Jacobi relations for the amplified system and giving rise to a group $\mathfrak{A}^{ij}$. This is then transformed so that $\mathfrak{A}^{ij}$ becomes the jth derivative of $\mathfrak{A}^{i0}$ and forms thus a differential group. The transformation is determined recursively by making use of the author's determination of all possible associative operations of rank 1 in the paper cited first. The differential group obtained is unique up to substitution, as above

T. Nakayama (Urbana, Ill.).

Segal, I. E., and von Neumann, John. A theorem on unitary representations of semisimple Lie groups. Ann. of Math. (2) 52, 509–517 (1950).

Démonstration du théorème suivant: Soit G un groupe de Lie connexe semi-simple dont aucune composante simple n'est compacte; soit $s\rightarrow U_s$ une représentation unitaire (de dimension infinie) de G, telle que l'anneau d'opérateurs engendré par les U_s soit un facteur; alors ce facteur n'est pas de classe (II_1). La démonstration procède comme suit. On suppose G simple; alors G contient un sous-groupe analytique isomorphe soit au groupe $x\rightarrow ax+b$, soit à un groupe résoluble à trois paramètres dont l'algèbre de Lie admet une base X, Y, Z telle que $[X,Y]=0$, $[X,Z]=X-aY$, $[Y,Z]=aX+Y$ (a réel non nul); on montre alors que toute représentation d'un tel sous-groupe dans un facteur de classe finie se réduit à l'identité; cela fait, une telle représentation de G se réduit à l'identité sur le sous-groupe en question de G, donc aussi sur les conjugués de ce sous-groupe, donc sur G tout entier puisque G est simple et connexe. La démonstration se simplifie quand G est complexe, car alors G contient toujours un sous-groupe isomorphe à $x\rightarrow ax+b$, ce qui dispense de considérer l'autre sous-groupe. Les auteurs terminent par une remarque sur les représentations unitaires mesurables d'un groupe localement compact G dans un espace de Hilbert H non nécessairement séparable; une telle représentation est somme directe (1) d'une représentation continue, (2) d'une représentation "singulière," i.e., telle que toute fonction de la forme (U_sx, y) soit nulle presque partout sur tout compact de G; si H est séparable, la composante singulière disparait. *R. Godement* (Nancy).

Itô, Seizô. Positive definite functions on homogeneous spaces. Proc. Japan Acad. 26, no. 1, 17–28 (1950).

Le but de cet article est d'étendre les principaux résultats de Gelfand et Raikov sur les fonctions de type positif [Rec. Math. [Mat. Sbornik] N.S. 13(55), 301–316 (1943); ces Rev. 6, 147] aux fonctions définies sur des espaces homogènes. Pour celà, soient G un groupe localement compact et Ω un espace homogène de groupe G, isomorphe à G/H où l'on suppose H compact; soit p_0 un point fixé de Ω (par exemple, l'image dans G/H de e); l'auteur définit une représentation unitaire du système (Ω, G, p_0) comme étant formée d'un espace de Hilbert H, dont un point ζ est donné à l'avance, d'une représentation unitaire $\sigma\rightarrow U(\sigma)$ de G dans H, enfin d'une application continue $p\rightarrow\zeta_p$ de Ω dans H telle que $\zeta_{p_0}=\zeta$, $U(\sigma)\zeta_p=\zeta_{\sigma(p)}$ pour $p\in\Omega$, $\sigma\in G$. Une telle représentation est dite cyclique si les $U(\sigma)\zeta$ engendrent H, et irréductible si la représentation U de G l'est. Enfin, une fonction $f(p,q)$ définie sur $\Omega\times\Omega$ est dite de type positif si $f(\sigma(p),\sigma(q))=f(p,q)$ et si $\iint f(p,q)x(p)\overline{x(q)}dp\,dq\geqq 0$ pour toute fonction x continue à support compact sur Ω (dp désigne la mesure invariante sur Ω); définition équivalente: $f(\sigma(p),\sigma(q))=f(p,q)$, et $f(\sigma(p_0),p_0)$ est de type positif sur G.

Si $\{H, U, \zeta\}$ est une représentation unitaire au sens défini plus haut, la fonction $f(p,q)=(\zeta_p,\zeta_q)$ est de type positif; on a ainsi une correspondance biunivoque entre fonctions de type positif sur Ω et représentations unitaires cycliques. L'auteur déduit de là que, comme dans le cas où $\Omega=G$, toute fonction de type positif mesurable et bornée coincide presque partout avec une fonction continue; d'autre part, les f continues de type positif telles que $f(p_0,p_0)\leqq 1$ forment dans l'espace $L^\infty(\Omega;dp)$ un ensemble convexe faiblement compact, dont les points extrémaux non nuls (fonctions "élémentaires") correspondent aux représentations irréductibles. On a donc comme dans le cas des groupes un théorème d'approximation par les combinaisons linéaires de telles fonctions. L'auteur montre aussi que, sur l'ensemble des f de type positif telles que $f(p_0,p_0)=1$, la convergence faible équivaut à la convergence uniforme sur tout compact (résultat dû à Raikov dans le cas des groupes). Enfin, on montre que toute fonction $f(p,q)$ continue et "invariante" sur $\Omega\times\Omega$ est approchable, uniformément sur tout compact, par des combinaisons de fonctions élémentaires de type positif. Un appendice montre comment les résultats précédents permettent de retrouver ceux de H. Weyl relatifs au cas ou G est compact. [Note du rapporteur: (1) Le cas où Ω est un espace de Riemann symétrique a été étudié plus en détail par Gelfand [Doklady Akad. Nauk SSSR. N.S. 70, 5–8 (1950); ces Rev. 11, 498]; (2) il est probable que, au moins lorsque G est séparable, on peut (c'est vrai dans le cas des groupes) représenter toute fonction de type positif par une intégrale étendue à l'ensemble des fonctions élémentaires.] *R. Godement* (Nancy).

Montgomery, Deane. Simply connected homogeneous spaces. Proc. Amer. Math. Soc. 1, 467–469 (1950).

As a consequence of E. Cartan's topological factorization of connected Lie groups in a compact subgroup and a Euclidean space, the author proves that a connected transitive Lie group in a compact manifold possesses a transitive compact subgroup. *H. Freudenthal* (Utrecht).

NUMBER THEORY

Pizá, Pedro A. Sur les puissances des nombres triangulaires. Mathesis 59, 145–155 (1950).

For integers n, c with $1\leqq c\leqq n$ the author introduces the symbol $[n:c]$ to designate the number $2^{c-1}\binom{2n}{c-1}/c$ and calls these quantities "Fermat coefficients." Letting T_c be the sum of the cth powers of the first $x-1$ triangular numbers, the author obtains the expansion

$$x^{2n+1}-x=2(2n+1)\sum_{c=1}^{n}[n:c]T_c,$$

and various identities are obtained from this. Various simple divisibility properties of the coefficients are also given.

H. W. Brinkmann (Swarthmore, Pa.).

Papademetrios, Ioannos. Concerning Fermat's numbers and Euclid's perfect numbers. Bull. Soc. Math. Grèce 24, 103–110 (1949). (Greek. English summary)

The author proves: If π is an odd prime which divides a number of the form $1+2^x$ and if $1+2^r$ is the smallest such number divisible by π, then x is an odd multiple of r. Using this lemma he then proves that if $r=2k\pi+1$ is a prime (π odd) which divides a number of the form $1+2^x$, then r also divides $1+2^{k\pi}$ for $k=1, 2$. Finally, a proof is given that there is no odd perfect number with less than four distinct prime factors. [Sylvester [Collected Mathematical Papers, v. 4, pp. 607–610 = C. R. Acad. Sci. Paris 106, 446–450 (1888)] has proved that no odd perfect number can have less than five distinct prime factors.] *T. M. Apostol.*

Blanuša, D. Une interprétation géométrique du crible d'Ératosthène. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 4, 201–202 (1949). (Croatian. French summary)

En joignant les points $A_r(0, a/r)$, $r=1, 2, \cdots$, aux points $B_s(s, 0)$, $s=2, 3, \cdots$, on obtient sur la droite $y=-a$ les points d'intersection $C_k(k, -a)$ avec $k=(r+1)s$, $r=1, 2, \cdots$; $s=2, 3, \cdots$, c'est à dire les points dont les abscisses sont tous les nombres naturels à l'exception de l'unité et des nombres premiers. *Author's summary.*

Gentile, Giovanni. Numeri primi in un intervallo particolare. Period. Mat. (4) 28, 130 (1950).

An elementary proof is given of the following theorem. Let $p_1=2$, $p_2=3$, $\cdots$ be the primes in order. Then for every integer n there are at least p_n-2 primes between p_n and $p_1\cdots p_n$. This is an improvement of a result of A. Lévy [Bull. Math. Élémentaires 15, 33–34, 81–82, 242–244 (1910)], whose theorem showed the existence of at least n such primes. *D. H. Lehmer* (Berkeley, Calif.).

Gut, Max. Eulersche Zahlen und grosser Fermat'scher Satz. Comment. Math. Helv. 24, 73–99 (1950).

The connection between the last theorem of Fermat and the Bernoulli numbers was indicated by Kummer more than a century ago. This is the basis of a number of criteria that have proved useful. The author shows that a similar connection exists between the Fermat problem and the Euler numbers E_n. As a parallel to a theorem of Mirimanoff, it is proved that if l is a prime the equation (1) $X^{2l}+Y^{2l}=Z^{2l}$ has no solution in rational integers X, Y, Z, prime to l if at least one of E_{l-3}, E_{l-5}, E_{l-7}, E_{l-9}, E_{l-11} is divisible by l [cf. Vandiver, Amer. J. Math. 62, 79–82 (1940); these Rev. 1, 200]. The familiar Kummer criteria involving the polynomials $\sum_{m=1}^{l} m^{n-1}t^m$ and Bernoulli numbers have parallel criteria involving the polynomials $\sum_{m=1}^{l}(2m-1)^{n-1}t^m$ and Euler numbers. *D. H. Lehmer* (Berkeley, Calif.).

Zakay, Shlomo. Leudesdorf's theorem in case of an even modulus. Riveon Lematematika 4, 35–37 (1950). (Hebrew. English summary)

The theorem in question has to do with the congruence properties of the sum of the reciprocals of the numbers less than and prime to a given modulus m [see Hardy and Wright, An Introduction to the Theory of Numbers, Oxford, 1938, p. 100]. When m is even, the proof is somewhat involved. The author suggests a modification of the proof based on a lemma dealing with sums of the type $\sum(n-t)^{-1}t^{-1}$ taken over the numbers t less than and prime to n.

D. H. Lehmer (Berkeley, Calif.).

Skolem, Th. An arithmetical property of the function

$$\sum_{n=0}^{\infty} \frac{x^n}{\prod_{i=0}^{\lambda} p_i^{\kappa_i(n)}},$$

where the p_i are natural primes and the $\kappa_i(n)$ polynomials with integral coefficients. Norske Vid. Selsk. Forh., Trondheim 22, no. 39, 183–187 (1950).

Mit einem kurzen elementaren Beweis wird gezeigt, dass die Funktion $f(x)=\sum_{n=0}^{\infty}\{x^n/\prod_{i=0}^{\lambda}p_i^{\kappa_i(n)}\}$, wobei $p_1, \cdots, p_\lambda$ Primzahlen und $\kappa_i(n)=a_in^2+b_in$ eine in n quadratische Form mit nichtnegativen, ganzrationalen a_i ($i=1, \cdots, \lambda$) und positivem a_1, sowie ganzrationalen b_i sei, für $x=0$ und $x=r=\pm1, \cdots, \pm l$, wobei r die zu $P=\prod_{i=1}^{\lambda}p_i^{2a_i}$ relativ primen Reste durchlaufe, im Körper der rationalen Zahlen linear unabhängige Werte annimmt; es ist also

$$c_0f(0)+\sum_{r=-l}^{+l}c_rf(r)=0$$

mit rationalen $c_0, c_{-l}, \cdots, c_l$ nur, wenn $c_0^2+\sum_{r=-l}^{l}c_r^2=0$. Damit wird in der Beantwortung der Frage nach den linear unabhängigen Werten $f(0), f(-l), \cdots, f(l)$ mit rationalen Koeffizienten eine Lücke geschlossen, da diese Tatsache bereits gezeigt ist [Skolem, in einer unveröffentlichten Arbeit] für jeden Grad der Polynome $\kappa_i(n)$, der >2 ist, und da diese Funktionswerte linear abhängig sind im Körper der rationalen Zahlen, falls alle Polynome $\kappa_i(n)$ vom ersten Grade sind. *T. Schneider* (Göttingen).

***Pollard, Harry. The Theory of Algebraic Numbers.** Carus Monograph Series, no. 9. The Mathematical Association of America, Buffalo, N. Y. (distributed by John Wiley and Sons, Inc., New York, N. Y.), 1950. xii+143 pp. \$3.00 (Members of The Mathematical Association of America may purchase one copy at \$1.75, order to be placed with the Secretary-Treasurer).

A clear and rigorous exposition of the elementary part of the theory of algebraic numbers, following classical lines (as by Hilbert), but in more accessible form and with modifications of some particular points. The theory is constructed by Dedekind's method of ideals (and not by the method of valuations). For the fundamental theorem of factorization two proofs are given: the classical Dedekind proof, and a proof by Noether-Artin theory. The discriminant of a field is defined as the minimum of the discriminants of its integral bases. Dedekind's ramification theorem is mentioned, but only the direct part (all ramified ideals divide the discriminant) is proved. For Dirichlet's theorem on units Dirichlet's proof is given, and not that of van der Waerden. The finiteness of the class number is established, but its calculation by $\zeta_k(s)$ is not studied. Galois theory and relative theory with respect to a basis field are not considered. The monograph contains also: a proof of the existence of transcendental numbers (Liouville numbers), determination of the integral bases and units of the quadratic fields, determination of integral bases and discriminants of the cyclotomic fields, and the proof of Fermat's last theorem for regular prime exponents. *M. Krasner.*

Holzer, L. Zur Klassenzahl in reinen Zahlkörpern von ungeraden Primzahlgrade. Acta Math. 83, 327–348 (1950).

Let R denote the field of rational numbers, let Ω be a field of the form $\Omega=R(m^{1/l})$, where l is an odd prime number and m is a rational integer. Necessary conditions are given

in order that the class number of Ω be divisible by l. The following further results may be mentioned: If, for some $s>0$, there exist $s+\frac{1}{2}(l-1)$ prime factors of m of the form $xl+1$, then at least s basis elements of the group of ideal classes of Ω have orders divisible by l. The same statement is true, if all prime divisors of m different from l are primitive roots mod l and if the number of these prime divisors is $s+\frac{1}{2}(l-1)$ in the case $m^{l-1}\not\equiv 1 \bmod l^2$, and $s+\frac{1}{2}(l+1)$ in the case $m^{l-1}\equiv 1 \bmod l^2$. If the class number of the field of the lth roots of unity is not divisible by l and if m is a prime number which is a primitive root mod l, then the class number of Ω is not divisible by l. *R. Brauer.*

***Jones, Burton W. The Arithmetic Theory of Quadratic Forms.** Carus Monograph Series, no. 10. The Mathematical Association of America, Buffalo, N. Y. (distributed by John Wiley and Sons, Inc., New York, N. Y.), 1950. x+212 pp. \$3.00 (Members of The Mathematical Association of America may purchase one copy at \$1.75, order to be placed with the Secretary-Treasurer).

This excellent monograph gives an introduction to the arithmetic parts of the theory of quadratic forms in self-contained form. It assumes only knowledge of the most elementary facts in the theory of numbers and the theory of matrices and, moreover, it is written in a very clear style. For these reasons it will make easy reading even for beginning students. Yet it leads up to the limits of what is known, taking into account recent work of Hasse, Pall, Ross, the author himself and others. In this respect it partly fills a gap in the existing literature, since no other exposition in book form of the more recent development of the arithmetic theory of quadratic forms exists. The general problem of the theory is that of the representation of quadratic forms in m variables by a given quadratic form in n variables ($1\leqq m\leqq n$) or, in the language of matrices, the problem of the representation of symmetric ($m\times m$)-matrices B by a given symmetric ($n\times n$)-matrix A in the form $B=X^TAX$ by means of an ($n\times m$)-matrix X (X^T is the transposed matrix of X). The elements of these matrices can be restricted to given fields or rings. The titles and contents of the chapters are as follows. I: Forms with real coefficients. Conditions for the possibility of the representation of one quadratic form by another. Automorphs. II: Forms with p-adic coefficients. Definition of p-adic numbers. Relation between rational representability modulo a prime power p^t of a matrix B by a matrix A and p-adic representability of B by A. The symbols of Hilbert and Hasse. Zero forms. Conditions for p-adic representability of B by A. Universal forms. III: Forms with rational coefficients. Equivalence. Reduced forms. The fundamental theorem of Hasse (a rational nonsingular form is a zero form if and only if it is a p-adic zero form for all primes p and a real zero form). Rational equivalence and rational representation. IV: Forms with coefficients in $R(p)$ (the ring of p-adic integers). Canonical forms. Representation of numbers by forms and equivalence of forms in $R(p)$. Representation of one form by another. Zero forms. Universal forms. Automorphs. Binary forms. V: Genera and semi-equivalence. Representation without essential denominator. Existence of forms with integral coefficients and having given invariants. VI: Representation by forms. Siegel's representation function. Asymptotic results. VII: Binary forms. Automorphs. Representation by binary forms. Ideals in a quadratic field. Correspondence between ideal classes and classes of quadratic forms. Composition of classes of ideals and classes of forms. Genera. Reduction. Class number. VIII: Ternary quadratic forms. Numbers represented by ternary genera. One minor remark should be made here about the treatment of p-adic numbers in chapter II. This treatment is incomplete since it does not introduce the notion of p-adic convergence. In some places, however (e.g., on p. 22), where this notion is actually needed in the proofs, the author then uses the rather vague expression that a sequence of numbers "generates" its limit. There is a collection of 95 (for the greater part very interesting) problems at the end of the book as exercises for the reader. *H. D. Kloosterman.*

Bateman, P. T., Chowla, S., and Erdös, P. Remarks on the size of $L(1,\chi)$. Publ. Math. Debrecen 1, 165–182 (1950).

Let (d/n) denote the Kronecker symbol,

$$L_d(1)=\sum_{n=1}^{\infty} n^{-1}(d/n),$$

similarly

$A=\limsup L_d(1)(\log\log d)^{-1}$ and $a=\liminf L_d(1)(\log\log d)^{-1}$ as $d\to\infty$; B and b are defined similarly as $d\to-\infty$. The authors prove that $A\geqq(18)^{-1}e^\gamma$, $B\geqq(18)^{-1}e^\gamma$, $a\leqq 3\pi^2e^{-\gamma}$, $b\leqq 3\pi^2e^{-\gamma}$, and announce without proof the stronger results $A\geqq\frac{1}{4}e^\gamma$, $B\geqq\frac{1}{4}e^\gamma$, $a\leqq\frac{3}{8}\pi^2e^{-\gamma}$, $b\leqq\frac{3}{8}\pi^2e^{-\gamma}$. ~~Slightly weaker~~ results were obtained by Chowla [Proc. London Math. Soc. (2) 50, 423–429 (1949); these Rev. 10, 285]. The proof is based on the sieve method of Linnik and Rényi [J. Math. Pures Appl. (9) 28, 137–149 (1949); these Rev. 11, 161]. The authors also give a slight numerical improvement of the classical inequality $L(1,\chi)<\log k$, where χ is any non-principal character mod k. The following misprints were communicated by one of the authors: p. 167, line 12, replace 5/4 by 7/4; p. 167, second line from bottom, replace $\sum_{n=1}^{n}$ by $\sum_{n=1}^{m}$; p. 171, fifth line from bottom, replace $\log(\frac{1}{2}\log\log x-\log\log\log x)$ by

$$\log(\tfrac{1}{2}\log\log x-2\log\log\log x);$$

p. 174, line 7, replace $p\epsilon\mathfrak{S}$ by $q\epsilon\mathfrak{S}$; p. 176, line 6, replace $b\epsilon\mathfrak{S}$ by $q\epsilon\mathfrak{S}$; p. 177, last line, replace $\frac{1}{2}b(\log x)1+\delta$ by $\frac{1}{2}b(\log x)^{1+\delta}$; Chowla [3] in the references, replace Proc. Nat. Acad. Sci. India by Proc. Nat. Inst. Sci. India. *H. Heilbronn* (Bristol).

Shapiro, Harold N., and Warga, Jack. On the representation of large integers as sums of primes. I. Comm. Pure Appl. Math. 3, 153–176 (1950).

Schnirelmann [Izvestiya Donskogo Politehničeskogo Instituta 14, 3–28 (1930)] proved the existence of a constant κ such that every large integer can be expressed as a sum of κ or less prime numbers. In the following years various authors gave explicit estimates for κ, the best one being Ricci's $\kappa\leqq 67$ [Ann. Scuola Norm. Super. Pisa (2) 6, 71–90, 91–116 (1937)]. These results have been superseded by Vinogradov's theorem that every large odd integer is a sum of three primes and hence $\kappa\leqq 4$ [Rec. Math. [Mat. Sbornik] (N.S.) 2, 179–194 (1937)]. Vinogradov, however, uses the most powerful tools of analytic number theory while the preceding papers are essentially elementary. The same holds true of the present paper in which $\kappa\leqq 20$ is proved. In fact, except for one point where the prime number theorem for arithmetic progressions is needed, this proof is completely elementary. It may conveniently be divided into three sections.

Let $N_2(y)$ denote the number of decompositions of y into a sum of two primes and put $w(y)$

$$=N_2(y)[\prod_{p>2}(1-(p-1)^{-2})(y/\log^2 y)\prod_{p|y,\,p>2}(p-1)/(p-2)]^{-1}.$$

The first part leads by means of A. Selberg's modification of the sieve method to (1) $\limsup w(y)\leqq 16$ [theorem 2.1]. Part II consists of the proof of

$$(2)\qquad \lim x^{-1}\sum_{y\leqq x,\,y\equiv 0\ (\mathrm{mod}\ 2)} w(y)=1$$

[cf. formula (3.3)]. It is based on ideas of Schnirelmann, Romanov, and Landau which have been used in a similar fashion in other pre-Vinogradov papers [cf. Heilbronn, Landau, and Scherk, Časopis Pěst. Mat. Fys. **65**, 117–140 (1936)]. Finally, comparison between (1) and (2) leads at once to $\liminf \sum_{y\leqq x,\,y\equiv 0\ (\mathrm{mod}\ 2),\,N_2(y)>0} x^{-1}\geqq 1/16$ [cf. theorem 3.1]. A well-known result on the Schnirelmann-sums of sets of integers then yields $\kappa\leqq 20$ [cf. Khintchine, Rec. Math. [Mat. Sbornik] **39**, 27–34 (1932)]. *P. Scherk.*

Mahler, Kurt. On the continued fractions of quadratic and cubic irrationals. Ann. Mat. Pura Appl. (4) **30**, 147–172 (1949).

Verfasser zeigt nach einer Methode von F. J. Dyson [Acta Math. **79**, 225–240 (1947); diese Rev. **9**, 412]: Ist ζ eine reelle algebraische Zahl $n$ten Grades und hat die Ungleichung $|p/q-\zeta|<q^{-\mu}$ unendlich viele Lösungen in rationalen Zahlen p/q, wobei der grösste Primfaktor von $q\geqq 1$ beschränkt ist, so folgt $\mu\leqq n^{\frac12}$. Daraus ergibt sich, dass der grösste Primfaktor von q_n, wenn mit q_n der $n$te Kettenbruchnäherungsnenner bezeichnet wird, mit n gegen Unendlich strebt, falls ζ eine reelle quadratische oder kubische Irrationalzahl bedeutet. Damit wird ein früheres Resultat des Verfassers [Akad. Wetensch. Amsterdam, Proc. Sect. Sci. **39**, 633–640, 729–737 (1936)] in einem spezielleren Falle verschärft. *T. Schneider* (Göttingen).

Davenport, H. On a theorem of Khintchine. Proc. London Math. Soc. (2) **52**, 65–80 (1950).

Khintchine [Rend. Circ. Mat. Palermo **50**, 170–195 (1926)] proved the existence of a positive absolute constant δ, such that for any real number α there exists at least one real number β with the property that $|\alpha x+\beta-y|>\delta/x$ for all integers $x>0$, y. Fukasawa [Jap. J. Math. **4**, 41–48 (1927)] gave a construction for β and a numerical estimate for δ, namely $\delta\geqq 1/457$ [see the bibliography in the reviewer's book, Diophantische Approximationen, Springer, Berlin, 1936, chapter VI]. The object of this paper is to give another arithmetical construction for β by means of the expansion of α in a semi-regular continued fraction by the method of the nearest integer. The author also deduces a better estimate for δ and he remarks that his pupil Prasad, by improving the method, has found a still better estimate, namely $\delta\geqq 3/32$. *J. F. Koksma* (Amsterdam).

Mullender, P. Simultaneous approximation. Ann. of Math. (2) **52**, 417–426 (1950).

The author deals with the following problems: (1) to give the lower bound of the numbers $c>0$ for which the simultaneous inequalities $|x|\geqq 1$, $|\alpha-y/x|<c^{\frac12}/|x|^{\frac32}$, $|\beta-z/x|<c^{\frac12}/|x|^{\frac32}$ for any real α and β have an infinity of integral solutions x, y, z; and (2) to give the lower bound of the numbers $c>0$ for which the simultaneous inequalities $X=\max(|y|,|z|)\geqq 1$, $|\alpha y+\beta z-x|<c/X^2$ for any real α and β have an infinity of integral solutions x, y, z. These problems still seem far from a solution, but several authors have given estimates for admissible values of c [Minkowski, Geometrie der Zahlen, Teubner, Leipzig-Berlin, 1910; Blichfeldt, Trans. Amer. Math. Soc. **15**, 227–235 (1914); Koksma and Meulenbeld, Nederl. Akad. Wetensch., Proc. **44**, 62–74 (1941); Mullender, ibid. **50**, 173–185 (1947); **51**, 874–884 (1948) = Indagationes Math. **9**, 136–148 (1947); **10**, 302–312 (1948); Davenport and Mahler, Duke Math. J. **13**, 105–111 (1946); these Rev. **2**, 253; **9**, 335; **10**, 285; **7**, 506]. By an ingenious method the author now improves all known estimates, proving that in both problems, each value of $c>2^7/3^4\cdot 3^{\frac12}\cdot 23^{\frac12}=1/2.400$ is admissible. This result is remarkable in view of a related problem for which Davenport and Mahler [loc. cit.] proved the value $2/23^{\frac12}=1/2.398$ to be the exact lower bound of c. By using results of Furtwängler [Math. Ann. **96**, 169–175 (1926); **99**, 71–83 (1928)] and the author [first reference cited] it is shown that the exact lower bound of c in any case is not less than $23^{-\frac12}$. *J. F. Koksma* (Amsterdam).

Mullender, P. On a theorem of Korkine-Zolotareff. Mathematica, Zutphen B. **13**, 23–27 (1944). (Dutch)

It was shown by Korkine and Zolotareff [Math. Ann. **6**, 366–389 (1873)] that the inequality

$$|(\alpha x+\beta y)(\gamma x+\delta y)|\leqq|\Delta|/A$$

has a solution in integers x,y not both zero for all real α, β, γ, δ with $\Delta=\alpha\delta-\beta\gamma\neq 0$ if and only if $A\leqq\sqrt{5}$. Mahler [same journal **8**, 57–61 (1939); these Rev. **1**, 39] showed that indeed there are infinitely many such solutions if $A\leqq\sqrt{5}$; it is this theorem for which the present author gives a new proof, using a method devised by Heilbronn [cf. Hardy and Wright, An Introduction to the Theory of Numbers, Oxford, 1938, pp. 389–390] for the original Korkine-Zolotareff theorem. *W. J. LeVeque.*

Fine, N. J. On the asymptotic distribution of the elementary symmetric functions (mod p). Trans. Amer. Math. Soc. **69**, 109–129 (1950).

Let $U_k^{(n)}$ denote the sum of the products, k at a time, of n numbers $x_1,\cdots,x_n$. Let p be a prime, and let $P_n(p,k,a)$ denote the number of sets of integers $x_1,\cdots,x_n$ (mod p) for which $U_k^{(n)}\equiv a$ (mod p). It is proved that $p^{-n}P_n(p,k,a)$ tends to a limit as $n\to\infty$, while p, k, a remain fixed. The greater part of the paper is devoted to a study of this limit, denoted by $P(p,k,a)$, the behaviour of which proves to be distinctly complicated. The only result which is obvious is that $P(p,k,a)=P(p,k,b)$ if $a\equiv bl^k$, $a\not\equiv 0$ (mod p). In particular, if k is relatively prime to $p-1$, the value of $P(p,k,a)$ is independent of a for $a\not\equiv 0$ (mod p). The author proves that if $k=tp^s$, where $0<t<p$ and $s\geqq 0$, then $P(p,k,a)$ has the same value, namely p^{-1}, for all a; but this seems to be exceptional, and the author suspects that these are the only values of k for which there is complete equality of distribution. When p is 2, it is proved that $P(p,k,1)=2^{-h}$ and $P(p,k,0)=1-2^{-h}$, where h is the number of nonzero digits in the expansion of k in the scale of 2. For general p, $P(p,k,0)\leqq 1-(1-p^{-1})^{-h}$, where h has a similar meaning for the scale of p. The author conjectures that for any p and k, the greatest value of $P(p,k,a)$ occurs when $a\equiv 0$ (mod p). The corresponding problem for $S_k^{(n)}=x_1^k+\cdots+x_n^k$ is also treated; here the answer is very simple, since it is proved that the corresponding limit is always p^{-1}. *H. Davenport* (London).

ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Terasaka, Hidetaka. Über linearen Kontinuen. Proc. Japan Acad. **22**, nos. 1–4, 61–68 (1946).

An L-continuum Λ is a linearly ordered space which is compact and a continuum in the topology in which segments are neighborhoods. If Λ is homeomorphic to each of its segments, it is called homogeneous. If α is an ordinal and if for each $\lambda<\alpha$, E_λ is a compact linearly ordered space containing at least two points, then $P_{\lambda<\alpha}E_\lambda$ is the space of all α-sequences $\xi=\{x_\lambda\}$, $\lambda<\alpha$, where $x_\lambda \epsilon E_\lambda$. Define $\xi=\eta$ if $x_\lambda=y_\lambda$ for all $\lambda<\alpha$, or, if α is a limit ordinal, if there is a $\mu<\alpha$ such that $x_\lambda=y_\lambda$ for $\lambda<\mu$, and $x_\mu=y_\nu=0$, $x_\nu=y_\mu=1$ for $\mu<\nu<\alpha$, where 0 and 1 ($\neq 0$) are the smallest and largest elements of E_λ, respectively. Define $\xi>\eta$ if $\xi\neq\eta$ and there exists $\mu<\alpha$ such that $x_\lambda=y_\lambda$ for $\lambda<\mu$, and $x_\mu>y_\mu$ (lexicographic order). If α is a limit ordinal, if F_λ contains only 0 and 1 for each $\lambda<\alpha$, and if $L^\alpha=P_{\lambda<\alpha}F_\lambda$, then L^α is an L-continuum.

Theorem. L^α is a homogeneous L-continuum if and only if α is a countable limit ordinal, and every initial segment of a well-ordered set of type α is smaller than its corresponding final segment (i.e., $\alpha=\beta+\gamma$ implies $\beta<\gamma$). In order to prove this the author derives results on products such as: If α is a countable limit ordinal and if all $E_\lambda=E$, a homogeneous L-continuum, then $P_{\lambda<\alpha}E_\lambda$ is a homogeneous L-continuum order-isomorphic to no subset of E; if $\alpha_1\geqq\alpha_2$, then $P_{n<3}L^{\alpha_n}$ is not homogeneous. *M. M. Day.*

Sudan, Gabriel. Remarque sur une note de Jordan. Acad. Roum. Bull. Sect. Sci. **30**, 321–323 (1948).

This is a simple proof of the nondenumerability of the points of an interval. Let $f(x)$ be bounded and nondecreasing on $[a, b]$. Then $f(x)$ has at least one point of continuity on $[a, b]$. Let $f(b)-f(a)=\nu\geqq 0$. Divide $[a, b]$ into four equal parts. For at least one of these parts $[a_1, b_1]$, $f(b_1)-f(a_1)<\frac{1}{2}\nu$. Continue to get $[a_n, b_n]$ with $b_n-a_n=(b-a)4^{-n}$, $f(b_n)-f(a_n)<2^{-n}\nu$. Then $f(x)$ is continuous at the point α defined by the infinite set of intervals $[a_1,b_1],[a_2,b_2],\cdots$. Now suppose the points of $[a, b]$ form a denumerable set $x_1, x_2, \cdots$. Let x be a point of $[a, b]$. Let x_n be the number of points of the set $x_1, x_2, \cdots$ on $[a, x)$ with subscript less than or equal to n. Let $F(x)=\sum_{n=1}^{\infty} n^{-1}x_n 2^{-n}$. The function $F(x)$ is nondecreasing and bounded and, therefore, has a point of continuity on $[a, b]$. But $F(x)$ is discontinuous at each of the points $x_1, x_2, \cdots$. Thus there is a contradiction, and the points of $[a, b]$ cannot be denumerable. *R. L. Jeffery* (Kingston, Ont.).

Denjoy, Arnaud. Le véritable théorème de Vitali. C. R. Acad. Sci. Paris **231**, 560–562 (1950).
Denjoy, Arnaud. Le théorème de Vitali. C. R. Acad. Sci. Paris **231**, 600–601 (1950).

The author points out that generalization of the Vitali theorem has proved difficult because the hypothesis of the theorem involves both a measure function and a distance function, while the conclusion is stated entirely in terms of the measure function. He arrives ultimately at the following generalization, using only a measure function: Let φ be a nonnegative, completely additive set function on a set of measurable subsets of the space U; let P and G be two families of measurable subsets ω and γ, respectively, such that each ω of P contains a γ, setting up relations $\omega(\gamma)$ and $\gamma(\omega)$. It is assumed: (1) that each point of the subset H of U is indefinitely covered by the sets ω, i.e., for each point M of H there exists a sequence ω_n containing M such that $\varphi(\omega_n)$ approaches zero; (2) there exist two numbers a and b with $1<a<b$ independent of ω such that if $\Omega(\gamma)$ is the union of the sets $\omega(\gamma')$ which intersect γ and for which $\varphi(\gamma')<a\varphi(\gamma)$, then $\bar{\varphi}(\Omega(\gamma))<b\varphi(\gamma)$, where $\bar{\varphi}$ may be the outer measure; (3) for each γ, the set $\rho(\gamma)$ of points of H not in γ and indefinitely covered by $\omega(\gamma')$ intersecting γ has φ-measure zero; (4) the union D of sets γ is of finite (outer) φ-measure. Under these hypotheses, there exists a denumerable set γ_n of disjoint sets chosen from G such that $\varphi(H-H\cdot\Gamma)=0$ where $\Gamma=\sum\gamma_n$. In the first note, it is assumed that the sets ω are open and the sets γ are closed, but the proof does not use these properties. *T. H. Hildebrandt.*

Kappos, Demetrios A. Über einen Satz der Theorie der Baireschen Funktionen und Borelschen Mengen. Math. Ann. **122**, 1–5 (1950).

Let V be a lattice in which every bounded countable subset admits an infimum and a supremum. Let V also be a topological lattice, in the sense that the operations $\vee$ and $\wedge$ of the lattice are continuous under o-convergence. For an arbitrary subset A of V, let A_σ and A_δ be respectively the sets of all suprema and infima of countable subsets of A. Let A_λ be the set of all o-limits of sequences of elements of A. Generalizing a theorem of Sierpiński concerning function-lattices [Fund. Math. **18**, 1–22 (1932)], the author proves that if W is a sublattice of V, then $W_\lambda=(W_\sigma)_\delta\cap(W_\delta)_\sigma$. *E. Hewitt* (Seattle, Wash.).

Kodaira, Kunihiko, and Kakutani, Shizuo. A non-separable translation invariant extension of the Lebesgue measure space. Ann. of Math. (2) **52**, 574–579 (1950).
Kakutani, Shizuo, and Oxtoby, John C. Construction of a non-separable invariant extension of the Lebesgue measure space. Ann. of Math. (2) **52**, 580–590 (1950).

The purposes of these two papers are similar; their methods, however, are different. The result of the first paper is, in fact, a special case of that of the second; the methods of the first paper are, however, relatively simple applications of well-known facts concerning topological groups, whereas the methods of the second paper are intricate set theoretic combinatorics. The results are as follows. (1) Lebesgue measure in the unit interval can be extended to a class of sets including the class of Lebesgue measurable sets so that the resulting measure is invariant under translations modulo 1 and so that its separability character is equal to c (the power of the continuum). (2) Lebesgue measure in the unit interval can be extended to a class of sets including the class of Lebesgue measurable sets so that the resulting measure is invariant under all one-to-one and Lebesgue measure preserving transformations of the unit interval onto itself and so that its separability character is equal to 2^c. The second assertion includes not only the first but also an earlier result of Kakutani [Proc. Imp. Acad. Tokyo **20**, 115–119 (1944); these Rev. **7**, 279]. That result, unlike the present one, placed no invariance requirements on the extended measure and employed the continuum hypothesis in its proof. *P. R. Halmos* (Chicago, Ill.).

Cafiero, Federico. Criteri di compattezza per le successioni di funzioni generalmente a variazione limitata. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 305–311 (1950).

A function f_n on the square $D: 0 \leqq x, y \leqq 1$ is "generally of bounded variation" if there exists a set $I_n \subset D$ with measure 0 such that if $V_x^{(n)}(\bar{y})$ is the variation of f_n on $y = \bar{y}$ when points of I_n are ignored and $V_y^{(n)}(\bar{x})$ is defined analogously, the functions $V_x^{(n)}$ and $V_y^{(n)}$ are summable on $[0, 1]$. The compactness theorem established is the following: Let $f_1, f_2, \cdots$ be a sequence of functions generally of bounded variation on D, and let there exist a constant k such that $\int_0^1 V_y^{(n)}(x)dx + \int_0^1 V_x^{(n)}(y)dy < k$ and $\int\int_D |f_n(x, y)| dxdy < k$. Then it is possible to choose from the sequence a subsequence which converges almost everywhere and also in mean of order one to a function which is also generally of bounded variation on D. *E. J. McShane.*

Popoff, Kyrille. Sur une extension de la notion de dérivée d'une fonction. II. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **39**, 221–250 (1943). (Bulgarian. French summary)

[For the first part see the same Annuaire, Livre I. **35**, 225–249 (1939); these Rev. **1**, 207.] Ce mémoire reproduit les résultats sur le même sujet contenus dans des mémoires précédents [Rend. Sem. Mat. Roma. (4) **3**, 162–170 (1939); Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1942**, no. 2; ces Rev. **1**, 207; **8**, 321] et quelques nouveaux résultats, montrant que la dérivée, généralisée au moyen du premier moment, jouit de presque tous les propriétés des dérivées classiques. *From the author's summary.*

Zahorski, Z. Sur la première dérivée. Trans. Amer. Math. Soc. **69**, 1–54 (1950).

The author defines certain classes $\mathfrak{M}_n$ of functions of a real variable, and discusses whether $\phi(x) \varepsilon \mathfrak{M}_n$ is a necessary or a sufficient condition for $\phi(x)$ to be the derivative of an everywhere differentiable continuous function. Some of the results have already been announced [Rec. Math. [Mat. Sbornik] N.S. **9**(51), 487–510 (1941); C. R. Acad. Sci. Paris **223**, 415–417 (1946); these Rev. **3**, 73; **8**, 141]. Let $\mathfrak{D}$ be the class of functions which are (finite or infinite) derivatives of everywhere differentiable continuous functions; $\mathfrak{B}$ the class of bounded functions; $\mathfrak{B}\mathfrak{D}$ their intersection. Let $\phi \varepsilon \mathfrak{J}$ if ϕ belongs to the first Baire class and satisfies the Darboux condition. It is well known that $\mathfrak{D} \subset \mathfrak{J}$, but $\mathfrak{D} \neq \mathfrak{J}$, and that $\mathfrak{B}\mathfrak{D} \supset \mathfrak{B}\mathfrak{A}$, where $\mathfrak{A}$ is the class of approximately continuous functions, but $\mathfrak{B}\mathfrak{D} \neq \mathfrak{B}\mathfrak{A}$.

Six classes, M_0 to M_5, of linear sets are defined. All sets concerned belong to F_σ and further: $E \varepsilon M_0$ if $x \varepsilon E$ implies that x is a two-sided limit point of E; $E \varepsilon M_1$ if $x \varepsilon E$ implies that x is a two-sided point of condensation of E; $E \varepsilon M_2$ if, for $x \varepsilon E$, every one-sided neighbourhood of x contains a part of E of nonzero measure; $E \varepsilon M_3$ if, for $E = \bigcup F_n$ (with F_n closed), there is a sequence $\{\eta_n\}$ $(0 \leqq \eta_n < 1)$, and for every $x \varepsilon F_n$ and every $c > 0$ there is $\epsilon(x, c)$ such that, when $hh_1 > 0$, $h/h_1 < c$, $|h + h_1| < \epsilon(x, c)$, one has $|E \cap (x+h, x+h+h_1)| > \eta_n |h_1|$; the definition of M_4 is that of M_3 with the added restriction $\eta_n > 0$; $E \varepsilon M_5$ if $x \varepsilon E$ implies x is a point of metric density 1 of E.

A function $\phi(x) \varepsilon \mathfrak{M}_n$, $n = 0, 1, \cdots, 5$, if, for every a, the sets $\{\phi(x) > a\}$, $\{\phi(x) < a\}$ belong to M_n. It is shown that $\mathfrak{J} = \mathfrak{M}_0 = \mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \mathfrak{M}_3 \supset \mathfrak{M}_4 \supset \mathfrak{M}_5 = \mathfrak{A}$. Other results obtained are: (i) $\mathfrak{D} \subset \mathfrak{M}_2$; (ii) if $\phi \varepsilon \mathfrak{D}$, then for every a, b we have $\{a < \phi(x) < b\} \varepsilon M_3$, and therefore, if $\phi \varepsilon \mathfrak{D}$ and is finite, $\phi \varepsilon \mathfrak{M}_3$; (iii) $\mathfrak{B}\mathfrak{M}_4 \supset \mathfrak{B}\mathfrak{D} \supset \mathfrak{B}\mathfrak{M}_5$, but there is no identity among these classes; (iv) $E \varepsilon M_4$ is a necessary and sufficient condition that there exist $\phi \varepsilon \mathfrak{B}\mathfrak{D}$ such that $0 < \phi(x) < 1$ for $x \varepsilon E$, $\phi(x) = 0$ for $x \bar{\varepsilon} E$, and therefore a class identical with $\mathfrak{B}\mathfrak{D}$ cannot be obtained by using any sub-class of M_4. It is shown that none of the classes $\mathfrak{M}_n$, except $\mathfrak{M}_5$, is additive. Finally, counter-examples are given proving (for example) the impossibility of expressing a general finite $\phi \varepsilon \mathfrak{D}$ as the sum of two functions of particular simpler types.

U. S. Haslam-Jones (Oxford).

Papoulis, Athanasios. On the strong differentiation of the indefinite integral. Trans. Amer. Math. Soc. **69**, 130–141 (1950).

If a function $f(x, y)$ of two real variables is integrable on an interval I the author writes

$$W(I) = \int\int_I f dxdy, \quad F(I) = \int\int_I |f| dxdy.$$

The strong upper and lower derivates $G_s^*(x, y)$, $G_{*s}(x, y)$ of any interval function $G(I)$ are the upper and lower limits of $G(I)/|I|$ as diam $I \to 0$ with $(x, y) \varepsilon I$. If these are equal, $G(I)$ is strongly differentiable at (x, y), and their common value is $G_s'(x, y)$. The main results proved are: (i) If $F(I)$ is strongly differentiable almost everywhere in I, then $W(I)$ is also strongly differentiable almost everywhere in I; (ii) there exists an integrable function such that $W(I)$ is strongly differentiable almost everywhere but $F_s'(x, y)$ exists nowhere in a set E of positive measure; in fact, $F_s^*(x, y) = +\infty$ everywhere in E. *U. S. Haslam-Jones* (Oxford).

Morse, Marston, and Transue, William. The Fréchet variation, sector limits, and left decompositions. Canadian J. Math. **2**, 344–374 (1950).

This paper, which is one of a series of recent contributions by the authors, contains many basic results on what may be called the theory of Fréchet variation of functions of several variables. It has been known for a long time that Fréchet-variation is less restrictive than Vitali-variation, and that its use is more natural and fruitful in connexion with the representation of bilinear functionals over product spaces, but a thorough-going investigation of the properties of Fréchet-variation and the application of those properties to the study of such topics in analysis as the convergence of multiple Fourier series where the notion of variation enters in an essential manner into the hypotheses of the principal theorems is due to the recent work of the authors.

The development of the full technique that is required for nontrivial applications is quite elaborate, and spread out over many papers but this paper marks an important step. Before one can review the main results, the notations have to be explained. Let R^μ denote a Cartesian space of points s with coordinates $s = [s^1, \cdots, s^\mu]$. Let $a = [a^1, \cdots, a^\mu]$ and $b = [b^1, \cdots, b^\mu]$ be two points in R^μ with $a^r < b^r$, $r = 1, \cdots, \mu$. Let J_r represent an interval for s^r chosen from the intervals (a^r, b^r), $(a^r, b^r]$, $[a^r, b^r)$, $[a^r, b^r]$. By a μ-interval in R^μ determined by a and b is meant a Cartesian product (*) $I^\mu = J_1 \times J_2 \times \cdots \times J_\mu$. When $J_r = (a^r, b^r)$ for each r, we shall denote I^μ by $I^\mu(a, b)$. The intervals $I^\mu(a, b]$, $I^\mu[a, b)$, and $I^\mu[a, b]$ are similarly defined. By an r-segment Q^r in R^μ is meant a Cartesian product of the form (*) in which $\mu - r$ of the J_i's are points and the remaining r of the J_i's are intervals as above $(r = 1, \cdots, \mu)$. The Fréchet variation of the function g over I^μ is denoted by $P^\mu[g, I^\mu]$. The assumptions $\hat{F}$ on g are that the variation $P^\mu(g)$ is finite and that there is at least one r-section, $r = 1, \cdots, \mu$, of the inter-

val I^{μ} parallel to each coordinate r-plane, on which $P^{r}(g)$ is finite.

The results are: (1) If g satisfies $\hat{P}$ over I^{μ}, g is bounded and L-measurable over I^{μ}. Its points of discontinuity lie on at most a countable number of $(\mu-1)$-planes parallel to the coordinate $(\mu-1)$-planes. (2) With each point a in μ-space let 2^{μ} sectors S_a be associated, being the respective open regions into which the μ-space R^{μ} is divided by the $(\mu-1)$-planes intersecting a parallel to the coordinate $(\mu-1)$-planes. When g satisfies conditions $\hat{P}$, $g(s)$ has a limit as $s\to a$ from any one of the open sectors S_a. These 2^{μ} limits may be different at a. (3) The function g, defined over I^{μ}, may be termed S-continuous ($\bar{S}$-continuous) of orientation type S_a ($\bar{S}_a$), if $g(s)\to g(a)$ as $s\to a$ from S_a ($\bar{S}_a$) for each point a of I^{μ}. Given a g which satisfies conditions $\hat{P}$ over an open μ-interval I^{μ}, and given a sector S_a invariant in orientation as a is varied, there exists a unique function g^{S} equal to g at the points of continuity of g and S-continuous ($\bar{S}$-continuous) of type S_a ($\bar{S}_a$). (4) If a is any point of I^{μ}, the Fréchet variation of g over an r-interval Q^{r} in a fixed open sector S_a with vertex at a tends to zero as the vertices of Q^{r} tend to a, provided g satisfies $\hat{P}$ over I^{μ}. (5) A counterpart for the classical second law of the mean in the form of an inequality for multiple integrals is established. These results indicate the points of similarity of the Fréchet variation with the Jordan variation. Applications are promised in further papers. *K. Chandrasekharan* (Bombay).

Ilieff, Ljubomir. Beitrag zum Problem von D. Pompeiu. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. (Math. Phys.) **44**, 309–316 (1948). (Bulgarian. German summary)

Pompeiu's problem is the following. Let $\mathfrak{D}$ denote a fixed schlicht domain in the plane, and let $\mathfrak{D}(r,\theta)$ denote its rigid transform by the vector $re^{i\theta}$. Determine the continuous functions $f(x,y)$ which satisfy the equation $\iint_{\mathfrak{D}(r,\theta)} f(x,y)\,dxdy=0$ for all (r,θ). The author proves that if $\mathfrak{D}$ is an isosceles triangle with vertex angle incommensurate with π, then $f(x,y)\equiv 0$ is the only solution to the problem. The same result extends to a trapezoid constructed from the preceding triangle by drawing a line through the triangle parallel to the base. *M. Reade* (Ann Arbor, Mich.).

Theory of Functions of Complex Variables

***Carathéodory, Constantin. Funktionentheorie. Band I.** Verlag Birkhäuser, Basel, 1950. 288 pp. 36.00 Swiss francs.

***Carathéodory, Constantin. Funktionentheorie. Band II.** Verlag Birkhäuser, Basel, 1950. 194 pp. 24.50 Swiss francs.

This is avowedly a textbook, with the selection of material dictated by the author's own interests and his desire not to include any topic requiring too lengthy preparation. The division into two volumes is merely a matter of convenience. There are seven main parts: Complex numbers, including a chapter on the non-Euclidean geometry of the circle, of which the author makes constant and effective use; Topological lemmas; Analytic functions; Generation of analytic functions by limiting processes; Special functions, including a chapter on the gamma function; Geometric function-theory (bounded functions, conformal mapping in general, mapping of the boundary); Triangle functions; and Picard's theorem. There are many novelties of exposition and improvements on the traditional presentation of particular topics. *R. P. Boas, Jr.* (Evanston, Ill.).

Robertson, M. S. Applications of a lemma of Fejér to typically-real functions. Proc. Amer. Math. Soc. **1**, 555–561 (1950).

The Faltung $F(z)=\sum_{n=1}^{\infty}a_nb_nz^n$ of two power series $f(z)=\sum_1^{\infty}a_nz^n$ and $g(z)=\sum_1^{\infty}b_nz^n$, regular in $|z|<1$ and with real coefficients, is studied. If $f(z)$ and $g(z)$ are typically-real, then $F(z)$ is typically-real for $|z|\leqq 2-3^{\frac{1}{2}}$, this being the best possible radius. If $f(z)$ and $g(z)$ are convex in the direction of the imaginary axis, then $F(z)$ has the same property in $|z|<1$. Other similar results on $F(z)$ are obtained. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Džrbašyan, M. M. On the completeness of the system of functions $\{z^{\lambda_n}\}$ on a circle with a radial cut. Doklady Akad. Nauk SSSR (N.S.) **74**, 173–176 (1950). (Russian)

Let $h(z)$ be a bounded, positive, integrable function in D: $|z|<1$ with a cut from 0 to -1. Let K be the class of functions regular in D and such that $\iint_D h(z)|f(z)|^2dxdy<\infty$. Theorem: $\{z^{\lambda_n}\}$, $\lambda_n>0$, $\lambda_n\uparrow\infty$, is closed in K, if (1) $h(z)\leqq Ah(z^{1/\alpha})$, $1\leqq\alpha\leqq\alpha_0$ (A, α_0 constants),

$$(2)\qquad H(\vartheta)=\sup_{0<r<1} h(re^{i\vartheta})$$

satisfies $v=-(1/2\vartheta)\ln H(\pi-\vartheta)\uparrow\infty$ as $\vartheta\to 0+$ and v has the inverse function $\vartheta=p(v)$, and

$$(3)\qquad \limsup_{R\to\infty}\int_1^R\{n(v)/v+p(v)-1\}v^{-1}dv=\infty,$$

where $n(v)=\sum_{\lambda_\nu<v}1$. The proof is based on the application of Carleman's inequality to the function

$$F(w)=\iint_D h(z)\overline{f(z)}z^{w}dxdy \qquad (f\epsilon K).$$

W. H. J. Fuchs (Ithaca, N. Y.).

Boas, R. P., Jr. Polynomial expansions of analytic functions. J. Indian Math. Soc. (N.S.) **14**, 1–14 (1950).

Expansion theorems are obtained in terms of Appell polynomials $\{p_n(z)\}$ (defined by the series $A(t)e^{zt}=\sum_0^{\infty}p_n(z)t^n$, $A(t)=\sum_0^{\infty}a_nt^n$, $a_0\neq 0$), corresponding to various classes of functions $A(t)$. Among these are: (i) $A(t)$ is entire and is "measurable" in the sense of Pfluger, and satisfies certain supplementary conditions (too long to state here) relative to a convex polygon P. In this case every $F(z)$ regular in P has a corresponding Appell expansion in P. (ii) $A(t)$ is entire of order 1 and minimum type (order 1 and mean type). Then every $F(z)$ regular at $z=0$ (regular in a sufficiently large neighborhood of $z=0$) has an Appell expansion valid in the circle of convergence of $F(z)$ (valid in some neighborhood of $z=0$). (iii) $A(t)$ is entire and satisfies certain conditions bearing on the function $M(r)=\max|A(t)|$ (for $|t|=r$). Then $F(z)$ regular in a sufficiently large neighborhood of $z=0$ has an Appell expansion in a neighborhood of $z=0$. The method of proof is made to depend on the representation of a function $F(z)$, regular in a bounded closed n-gon, as a sum of n exponential transforms. (In particular, this provides shorter proofs of (ii) than were given by the reviewer [Duke Math. J. **3**, 593–609 (1937)].) There are also some remarks, and a theorem, pointing to the application of the same method (of exponential transforms) to expansions in terms of sequences of polynomials of "type zero." *I. M. Sheffer* (State College, Pa.).

Boas, R. P., Jr. Differential equations of infinite order. J. Indian Math. Soc. (N.S.) 14, 15–20 (1950).

The equation $\sum_{k=0}^{\infty} a_k F^{(k)}(z) = G(z)$ ($G(z)$ regular in certain regions) is shown to have analytic solutions under various assumptions on the function $A(t) = \sum_0^{\infty} a_k t^k$ ($a_0 \neq 0$). The classes of functions $A(t)$ considered are essentially those considered by the author in the paper reviewed above, and the proofs make use of the exponential transforms of that paper. *I. M. Sheffer* (State College, Pa.).

***Shen, Y. C., and Leng, S. M. Certain Hermitian forms and their related problems.** National Peking University Semi-Centennial Volume, Mathematical, Physical and Biological Series, pp. 31–44, 1948.

The authors deal with the positive Hermitian form $H_n(z; \bar{z}) = \sum_{i,j=1}^{n} A_{ij}(1-\alpha_i \bar{z})^{-1}(1-\bar{\alpha}_j z)^{-1}$, where $A_{ij} = E_{ij}/E_n$, which plays a part in interpolation to analytic functions by rational functions [Y. C. Shen, Trans. Amer. Math. Soc. 60, 12–21 (1946); these Rev. 8, 20]. The α_i's, $i = 1, \cdots, n$, are given, $|\alpha_i| < 1$; E_n is the n-rowed permanent

$$|1/(1-\alpha_i \alpha_j)|_n^+ = \sum \{(1-\alpha_1 \bar{\alpha}_{r_1}) \cdots (1-\alpha_n \bar{\alpha}_{r_n})\}^{-1},$$

where $(r_1, r_2, \cdots, r_n)$ runs through all the permutations of $(1, 2, \cdots, n)$; E_{ij} is the $(n-1)$-rowed permanent derived from E_n by striking the ith row and jth column. They prove that (a) $H_n(z; \bar{z})$ is strictly subharmonic (i.e., $\partial^2 H_n/\partial x^2 + \partial^2 H_n/\partial y^2 > 0$) in every finite region not containing any of the points $1/\alpha_i$ in its interior, (b) if the α_i's lie on a circular arc orthogonal to the unit circle C (e.g., on a diameter of C), then $H_n(z; \bar{z})$ is not smaller or not greater than $\sum_1^n (1-|\alpha_i|^2)|1-\bar{\alpha}_i z|^{-2}$ for $|z| < 1$ or $|z| > 1$, respectively; there is equality only when $\alpha_1 = \alpha_2 = \cdots = \alpha_n$ and, trivially, when $z = \infty$ or $1/\bar{\alpha}_i$, $i = 1, \cdots, n$. The result (b) is deduced from properties of the Hermitian form

$$h_n(z; \bar{z}) = \sum A_{ij} z_i \bar{z}_j (1-\alpha_i \bar{\alpha}_j)^{-1};$$

the result (a) from theorem 1. The Hermitian $\sum A_{ij} z_i \bar{z}_j$ is nonnegative, its rank r is equal to the number of distinct points among $\alpha_1, \cdots, \alpha_n$. If $\alpha_1, \cdots, \alpha_r$ ($r \leqq n$) are distinct, then the r-rowed principal minor standing at the upper left hand corner of the determinant $|A_{ij}|_n$ is positive.

The proof is based on the expansion of the permanent $E_n(\alpha, \beta; t) = |(1-\alpha_i \beta_j t)^{-1}|_n^+$ into the power series $n! + a_1 t + a_2 t^2 + \cdots$, $|t| \leqq 1$; the values $a_1, a_2, \cdots$ are computed. Finally the theorem is applied to $h_n(z; \bar{z})$, an independent proof is given for the result, and the characteristic equation $F_n(\lambda) = 0$, associated with the matrix of $h_n(z; \bar{z})$, is reduced to a fairly simple and yet interesting form. *H. Kober* (Birmingham).

Shah, S. M. The maximum term of an entire series. VI. J. Indian Math. Soc. (N.S.) 14, 21–28 (1950).

[For part V see the same J. 13, 60–64 (1949); these Rev. 11, 508]. Let $f(z)$ be an entire function, $M(r)$ its maximum modulus, $\mu(r)$ the maximum term in its power series, $\nu(r)$ the rank of the maximum term. The author denotes the upper and lower limits of $\log\log\log M(r)/\log\log r$ by L and l and shows that

$$\limsup \frac{\log M(r)}{\nu(r)} \frac{\log\log M(r)}{\log r} \geqq 1/l$$

and that the lim inf of the same expression does not exceed $1/L$ if $\log M(r) \sim \log \mu(r)$. Additional theorems and examples indicate directions in which these results can and cannot be extended. *R. P. Boas, Jr.* (Evanston, Ill.).

***Walsh, J. L. The Location of Critical Points of Analytic and Harmonic Functions.** American Mathematical Society Colloquium Publications, Vol. 34. American Mathematical Society, New York, N. Y., 1950. viii+384 pp. $6.00.

Les points critiques d'une fonction analytique sont les points où la dérivée de la fonction s'annule; ceux d'une fonction harmonique (de deux variables) sont ceux où les deux dérivées partielles s'annulent. Comme l'auteur le reconnaît lui-même dans sa préface, le titre de l'ouvrage ne correspond pas exactement à son contenu, et il ne s'agit nullement d'un exposé d'ensemble visant à donner une revue complète de tous les résultats connus sur la théorie des points critiques; et même si on le compare au livre récent de Marden [The Geometry of the Zeros of a Polynomial . . . , Amer. Math. Soc., New York, 1949; ces Rev. 11, 101], qui lui-même ne vise pas à être complet, on constate que la substance des 82 premières pages de ce dernier dépasse déjà nettement celle du présent ouvrage. L'immense majorité des résultats de celui-ci se rapporte en effet à un unique problème: déterminer des régions ne contenant aucun point critique, en supposant connues certaines propriétés de la fonction envisagée (par exemple la répartition de ses zéros ou de ses pôles dans certaines régions, s'il s'agit d'une fraction rationnelle, ou la position de ses courbes de niveau, s'il s'agit d'une fonction harmonique). En outre, ces résultats sont, à quelques très rares exceptions près, obtenus par application des deux remarques élémentaires suivantes: une relation $\sum_i z_i = 0$ entre nombres complexes est impossible si: (1) ou bien tous les z_i sont dans un même demi-plan, et ne sont pas tous sur la frontière du demi-plan; (2) ou bien il existe un nombre k tel que $|\sum_{i \leq k} z_i| < |\sum_{i > k} z_i|$.

Appliqués à la dérivée logarithmique $\sum_k e_k/(z-a_k)$ d'un polynôme ou d'une fraction rationnelle, ces deux principes donnent, comme on sait, les théorèmes classiques de Gauss-Lucas et de Jensen pour les polynômes, le théorème de Bôcher pour les fractions rationnelles (cas où on suppose tous les zéros de la fraction dans un cercle et tous ses pôles dans un autre cercle ne rencontrant pas le premier), et plus généralement, les "théorèmes des deux cercles" de l'auteur pour les polynômes et fractions rationnelles, tels qu'ils sont exposés dans les chapitres cités du livre de Marden. Mais non content de reprendre cette exposition avec un luxe extraordinaire de détails, l'auteur s'attache à gonfler démesurément son ouvrage d'une foule interminable de corollaires, cas particuliers ou variantes des mêmes thèmes; par exemple, un chapitre (V) est consacré au cas des fractions rationnelles possédant une certaine "symétrie" dans la disposition des zéros et des pôles. Mais ce n'est pas tout, car parvenu à la fin des considérations sur les polynômes et fractions rationnelles [à la p. 216], on recommence le déroulement des mêmes conséquences, en remplaçant la somme finie $\sum_i z_i$, soit par une série (ce qui donne un chapitre sur les points critiques de certaines fonctions analytiques), soit par une intégrale (ce qui fournit matière à trois chapitres sur les fonctions harmoniques). Certains des cas particuliers examinés (mais non la majorité) donnent des énoncés assez élégants, comme par exemple ceux où interviennent des polygones non euclidiens comme région délimitant les points critiques des fractions rationnelles de la forme $\prod_i (z-a_i)/(1-\bar{a}_i z)$. Mais de façon générale, on peut se demander à quoi répond la publication d'un tel livre. L'auteur signale lui-même dans sa préface qu'il n'a pas en vue l'obtention de propriétés utiles au calcul numérique; comme on n'aperçoit pas par ailleurs d'autre théorie mathématique où tout celà pourrait

rendre le moindre service, on aboutit assez naturellement à la conclusion qu'il est difficile de considérer cette théorie comme autre chose qu'une série d'exercices élémentaires de calcul différentiel; et si on peut déjà porter ce jugement sur les théorèmes "fondamentaux" cités ci-dessus, à plus forte raison cela s'applique-t-il aux innombrables corollaires de l'auteur, dont la plupart n'ont même pas le mérite d'un énoncé simple et frappant. Ce n'est pas à dire qu'un ouvrage de ce genre ne puisse avoir son intérêt, par exemple dans l'enseignement des premières années de l'université. Mais on voit mal à quel titre il figure dans la collection des "Colloquium Publications," où sont parus tant de volumes consacrés aux problèmes les plus profonds et les plus difficiles des mathématiques contemporaines. *J. Dieudonné.*

de Bruijn, N. G. The roots of trigonometric integrals. Duke Math. J. 17, 197–226 (1950).

It is not possible to give a complete account of the many results contained in this paper. The main results are the following. (1) Let $f(t)$ be an even nonconstant entire function of t, $f(t)\geqq 0$ for real t, and such that $f'(t)$ can be approximated, uniformly in every bounded domain, by polynomials all of whose zeros are purely imaginary. (In other words, $f'(t)=\exp(\gamma t^2)g(t)$, where $\gamma\geqq 0$ and $g(t)$ is an entire function of genus $\leqq 1$ with purely imaginary zeros only.) Then $\Psi(z)=\int_{-\infty}^{\infty}\exp\{-f(t)\}e^{izt}dt$ has real zeros only. This result contains several results of Pólya as special cases [see, for example, J. Reine Angew. Math. 158, 6–18 (1927)]. (2) Let $P(t)=\sum_{-N}^{N}p_ne^{nt}$, where $\Re p_N>0$, $p_{-n}=\bar{p}_n$ for $n=0, 1, \cdots$; let $q(x)$ be regular in the closed sector $-\frac{1}{2}\pi-\arg p_N\leqq N\arg x\leqq\frac{1}{2}\pi-\arg p_N$ except possibly at 0 and at ∞, where $q(x)$ may have poles, and let $q(x)$ be real on the part of $|x|=1$ in the sector. Set $q(e^t)=Q(t)$. Then all but a finite number of the zeros of $\Phi(z)=\int_{-\infty}^{\infty}\exp\{-P(t)\}Q(t)e^{izt}dt$ are real.

In order to arrive at a proof of (2), the asymptotic behavior of $\Phi(z)$ for large $|z|$ is studied in some detail. The author concludes that $\Phi(z)$ has only a finite number of zeros outside any strip $|\Im z|\leqq\epsilon$ $(\epsilon>0)$. Together with an auxiliary result this is sufficient to show that $\Phi(z)$ has only a finite number of nonreal zeros. A proof of (1) is now obtained as follows. By (2), the zeros of $\Phi(z)=\int_{-\infty}^{\infty}\exp\{-P(t)\}e^{izt}dt$ lie in some strip $|\Im z|\leqq\Delta$. Now let $P'(t)$ have purely imaginary zeros only. If $\Delta>0$, by an auxiliary result, the zeros of $\Phi_1(z)=\int_{-\infty}^{\infty}\exp\{-P(t)\}P'(t)e^{izt}dt$ lie in a narrower strip $|\Im z|\leqq\Delta_1<\Delta$. However, $\Phi_1(z)=iz\Phi(z)$, so that the zeros of $\Phi(z)$ must be real. By letting $P'(t)\to f'(t)$, $P(t)\to f(t)$, uniformly in every bounded domain, (1) follows. Several miscellaneous results give information on the zeros of trigonometric integrals which are similar to the integrals (*) and (**) connected with the Riemann and Ramanujan zeta-functions, $\zeta(s)$ and $F(s)=\sum\tau(n)n^{-s}$: (*) $\Xi(2z)=\int_{-\infty}^{\infty}\varphi(t)e^{izt}dt$, where

$$\Xi\{i(\tfrac{1}{2}-s)\}=\xi(s)=\tfrac{1}{2}s(s-1)\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s),$$

$$\varphi(t)=\sum_{n=1}^{\infty}(2n^4\pi^2e^{9t/4}-3n^2\pi e^{5t/4})\exp(-n^2\pi e^t);$$

and (**) $\Xi^{\sim}(z)=\int_{-\infty}^{\infty}\varphi^{\sim}(t)e^{izt}dt$, where

$$\Xi^{\sim}(is)=(2\pi)^{-s-6}\Gamma(s+6)F(s+6),$$

$$\varphi^{\sim}(t)=\exp(-2\pi\cosh t)$$
$$\times\prod_{\nu=1}^{\infty}[\{1-\exp(-2\pi\nu e^t)\}\{1-\exp(-2\pi\nu e^{-t})\}]^{12}.$$

The Riemann hypothesis is that all the zeros of (*) are real; there exists a corresponding hypothesis about the zeros of (**). The following results of the author in this connection are characteristic. (3) If $\lambda>0$, $\mu\geqq 0$, $k=1$ or 2, then $\int_{-\infty}^{\infty}\exp(-\lambda\cosh t)(\mu+\cosh^k t)e^{izt}dt$ has real zeros only. (4) If $\lambda>0$, $0<\delta<\frac{1}{2}\pi$, n a positive integer, then there exists a number $\Delta=\Delta(\lambda, \delta, n)>0$ such that the zeros of $\int_{-\infty}^{\infty}\exp(-\lambda\cosh t)f(\cosh t)e^{izt}dt$ lie in the strip $|\Im z|\leqq\Delta$ for any real polynomial $f(y)$ of degree n whose zeros lie in the sector $\frac{1}{2}\pi+\delta\leqq\arg y\leqq\frac{3}{2}\pi-\delta$. (5) If $\lambda>0$ and if $f(y)=\prod_{\mu=1}^{\infty}(1+c_\mu y)$, $c_\mu\geqq 0$, $\sum c_\mu(1+c_\mu)^{-1}\leqq 2\lambda$, then

$$\int_{-\infty}^{\infty}\exp(-\lambda\cosh^2 t)f(\cosh t)e^{izt}dt$$

has real zeros only. *J. Korevaar* (Lafayette, Ind.).

Komatu, Yûsaku. On Robin's constant and a distortion theorem. Kōdai Math. Sem. Rep. 1950, 37–39 (1950).

For a domain B whose Green's function is $G(z, \zeta)$, the Robin constant $\gamma(\zeta)$ is defined by

$$G(z, \zeta)+\log|z-\zeta|=\gamma(\zeta)+O(|z-\zeta|).$$

M. Schiffer [Amer. J. Math. 68, 417–448 (1946), p. 442; these Rev. 8, 325] proved that

$$2(G(z, \zeta)+\log|z-\zeta|)\geqq\gamma(z)+\gamma(\zeta).$$

If B is simply connected, this inequality implies the Loewner distortion theorem that $|f'(w)|\leqq|w|^2/(|w|^2-1)$ is valid for any function $f(w)$ schlicht in $|w|>1$ and normalized at ∞ such that $f(w)=w+\sum_{n=0}^{\infty}c_nw^{-n}$ $(|w|>1)$. In this paper the author shows by elementary means that, conversely, the Loewner distortion theorem implies the Schiffer inequality for simply connected domains. From this he shows that the simply connected extremal domains for which equality holds in the Schiffer inequality (for $z\neq\zeta$) consist of the whole plane slit along circular arcs. These extremal domains and the point at which equality holds are determined explicitly.

G. Springer (Cambridge, Mass.).

Komatu, Yûsaku, and Nishimiya, Han. On distortion in schlicht mappings. Kōdai Math. Sem. Rep. 1950, 47–50 (1950).

Consider the family of functions $\{w(z)\}$ regular and schlicht in $|z|<1$ and normalized such that $w(0)=0$ and $w'(0)=1$. The spherical derivative of $w(z)$ is defined as $Dw(z)=|w'(z)|/(1+|w(z)|^2)$. Using the classical distortion theorems of Koebe and Bieberbach and of R. Nevanlinna, the authors obtain a distortion theorem giving upper and lower bounds on $Dw(z)$ in terms of $r=|z|$, which hold when $r\leqq r^*=\frac{1}{2}(5^{\frac{1}{2}}-1)$. These estimates are precise in the sense that there exist functions for which the bounds are attained. For the remaining range, $r^*<r<1$, he derives upper and lower bounds which are not precise. The precise bounds are given by very simple rational functions of r for $r\leqq r^*$ but are not analytic in r for the whole range $r<1$. He then considers the Riemann sphere above the w-plane and finds precise estimates in terms of $r_1=|z_1|$ and $r_2=|z_2|$ for the spherical distance $\delta(w(z_1), w(z_2))$ between the two points on the sphere which project stereographically into $w(z_1)$ and $w(z_2)$, these precise estimates holding for $r_1\leqq r^*$ and $r_2\leqq r^*$. He finally obtains simpler bounds for D and δ when he restricts the class $\{w(z)\}$ by further requiring that the image of $|z|<1$ under $w(z)$ be a convex domain.

G. Springer (Cambridge, Mass.).

Ozawa, Mitsuru. Some remarks on conformal mapping of multiply connected domains. Kōdai Math. Sem. Rep. 1950, 1–2 (1950).

The author states a result concerning the zeros of Bergman's kernel function which is unfortunately false. The error arises from an incorrect formula for the kernel of the region interior to a level curve of the Green's function. A second portion of the paper is devoted to the relation between the harmonic measures and certain canonical slit mappings. *P. R. Garabedian.*

Ozawa, Mitsuru. On bounded analytic functions and conformal mapping. I. Kōdai Math. Sem. Rep. 1950, 33–36 (1950).

The author establishes by the method of contour integration the estimates for a bounded analytic function $f(z)$ in a multiply-connected region which are usually based on the maximum principle for the harmonic function $\log |f(z)|$. Applications of the estimates are made.
P. R. Garabedian (Stanford University, Calif.).

Ozawa, Mitsuru. Some canonical conformal maps and representations. Kōdai Math. Sem. Rep. 1950, 51–52 (1950).

The author expresses a radial slit mapping in which one of the slits extends to the point at infinity in terms of the Bergman kernel function. *P. R. Garabedian.*

Lavrent'ev, M. A. A fundamental theorem of the theory of quasi-conformal mappings of two-dimensional regions. Amer. Math. Soc. Translation no. 29, 57 pp. (1950).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 513–554 (1948); these Rev. 11, 650.

Matsumura, Yoshimi. On modified bent-functions and Phragmén-Lindelöf's principle. Osaka Math. J. 2, 33–41 (1950).

[The author's name was misprinted Matsumuma in the original.] The author's object is to establish certain known results concerning the Phragmén-Lindelöf principle. His fundamental inequality (1.4) is not valid. (Example: $\exp(az+z^3)$, a real.) *M. Heins* (Providence, R. I.).

***Birkhoff, George D. The matrix in modern complex analysis.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 173–193, Rome, 1943.

The author discusses in a general way the role of matrices in the theory of functions of a complex variable, with especial reference to their use in linear difference, differential, and certain other types of functional equations in the complex domain. The major part of the developments in this paper have been utilised previously by the author in a concrete way [Ann. Inst. H. Poincaré 9, 51–122 (1939); these Rev. 8, 201]. Among the topics considered are problems of the type of the converse Riemann problem, the notion of equivalence of matrices of functions analytic except for a singular point, factorization of polynomial and entire matrices, the role of equivalence in the unification of the theory of linear differential, difference, and q-difference equations. The envisaged field of investigation should still be considered open. *W. J. Trjitzinsky* (Urbana, Ill.).

Nehari, Zeev. A class of domain functions and some allied extremal problems. Trans. Amer. Math. Soc. 69, 161–178 (1950).

Let D denote a region of the complex plane bounded by n closed analytic curves $\Gamma_1, \cdots, \Gamma_n$ $(\Gamma=\bigcup_1^n\Gamma_i)$ and let λ denote a function positive and continuous on Γ. Let ζ denote a generic point of D. The author establishes the existence of domain functions $K_\lambda(z,\zeta)$, $L_\lambda(z,\zeta)$ satisfying the following conditions: $K_\lambda(z,\zeta)$ and $L_\lambda(z,\zeta)-[2\pi(z-\zeta)]^{-1}$ are regular in D; $|K_\lambda(z,\zeta)|$ is continuous in $\bar{D}$, $|L_\lambda(z,\zeta)|$ is continuous in $\bar{D}-C_\epsilon$, where C_ϵ is a small open circle centered at ζ; K_λ and L_λ satisfy $\lambda(z)[K_\lambda(z,\zeta)]^*ds=i^{-1}L_\lambda(z,\zeta)dz$, $z\epsilon\Gamma$, $|dz|=ds$. These properties determine K_λ and L_λ uniquely. The domain functions K_λ and L_λ constitute a generalization of domain functions due to P. R. Garabedian [same Trans. 67, 1–35 (1949); these Rev. 11, 340]. The proof is based upon the study of auxiliary harmonic functions. The functions K_λ and L_λ are constructed with the aid of the domain functions used by Garabedian. A number of extremal results are obtained with the aid of the introduced domain functions. Among these the following are typical: (1) If g is analytic and single-valued in D and $g(\zeta)=1$ and if g is suitably restricted in its behavior near Γ,

$$\int_\Gamma \lambda(z)|g(z)|^2ds \geqq \int_\Gamma \lambda(z)|M(z)|^2ds,$$

where $M(z)=K_\lambda(z,\zeta)/L_\lambda(z,\zeta)$. Equality holds only for $g\equiv M$. (2) Let B_λ denote the class of single-valued analytic functions in D which vanish at ζ and satisfy $\lim\sup|f|\leqq\lambda$ at all points of the boundary. Then

$$|f'(\zeta)|\leqq F'(\zeta)=2\pi K_\mu(\zeta,\zeta),$$

where $\mu\lambda=1$ and $F(z)=K_\mu(z,\zeta)/L_\mu(z,\zeta)$. If $L_\mu(z,\zeta)\neq 0$ for $z\epsilon D$, $F\epsilon B_\lambda$ and the inequality is sharp.

The author wishes to point out that the word "radial" is to be omitted on p. 161, line 7, and p. 174, line 12.
M. H. Heins (Providence, R. I.).

Carathéodory, C. Bemerkung über die Definition der Riemannschen Flächen. Math. Z. 52, 703–708 (1950).

Corresponding to the various ways of setting up the concept of a Riemann surface, there are various ways to prove the fundamental theorem in the theory of uniformization of analytic functions of a complex variable. The author compares the approaches proposed and studied by H. Weyl [Die Idee der Riemannschen Fläche, Teubner, Leipzig-Berlin, 1913], the reviewer [Acta Litt. Sci. Szeged 2, 101–121 (1925)], and van der Waerden [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 93, 147–160 (1941); these Rev. 11, 22], with particular emphasis upon the so-called "welding processes" needed in the course of the proof of the fundamental mapping theorem. *T. Radó.*

Florack, Herta. Reguläre und meromorphe Funktionen auf nicht geschlossenen Riemannschen Flächen. Schr. Math. Inst. Univ. Münster, no. 1, 34 pp. (1948).

The theorems of Mittag-Leffler and Weierstrass are extended to arbitrary open Riemann surfaces. Also it is shown that every open Riemann surface covering the extended complex plane is the maximal domain for some analytic function. Essential use is made of the recent work of Behnke and Stein [Math. Ann. 120, 430–461 (1949); these Rev. 10, 696]. *M. Heins* (Providence, R. I.).

Wittich, Hans. Über den Einfluss algebraischer Windungspunkte auf die Wachstumsordnung. Math. Ann. 122, 37–46 (1950).

The author studies certain parabolic simply connected Riemann surfaces whose branch-points lie over $0, 1, \infty$. The question is to what extent the algebraic branch-points influence the growth of the meromorphic uniformizing func-

tion. If there is only one logarithmic branch-point the order is at least $\frac{1}{2}$, by the Denjoy theorem, and if there are only a finite number of algebraic branch-points this is known to be the precise order. The interest centers around what happens when infinitely many algebraic branch-points are introduced according to certain prescribed patterns. In this way it is possible to raise the order; it is also possible to make the defect which corresponds to a logarithmic branch-point equal to 0, and it is possible to make one of the ramification indices maximal. The method makes familiar use of conformal and quasiconformal mappings of the fundamental polygon in the modular plane, but many refinements are introduced. *L. Ahlfors* (Cambridge, Mass.).

Stein, Karl. Primfunktionen und multiplikative automorphe Funktionen auf nichtgeschlossenen Riemannschen Flächen und Zylindergebieten. Acta Math. 83, 165–196 (1950).

If z and ζ are variables on an algebraic Riemann surface R, a prime function on R is defined as an analytic function of the two variables z and ζ which vanishes on the "diagonal manifold" $z=\zeta$, and only there. These prime functions were used by Weierstrass for the decomposition of algebraic functions into prime factors. In the present paper, both the definition of a prime function and the Weierstrass decomposition theorem are generalized to the case of open Riemann surfaces. *Z. Nehari* (St. Louis, Mo.).

Rothstein, Wolfgang. Ein neuer Beweis des Hartogsschen Hauptsatzes und seine Ausdehnung auf meromorphe Funktionen. Math. Z. 53, 84–95 (1950).

Die Arbeit befasst sich mit dem folgenden Hartogsschen Hauptsatze: "Die Funktion $f(w_1, \cdots, w_n, z)$ sei in $\{|w_1|<1, \cdots, |w_n|<1, |z|<1\}$ regulär (meromorph). Ferner möge für jedes $(c_1, \cdots, c_n)$ aus $\{|w_1|<1, \cdots, |w_n|<1\}$ die Funktion $f(c_1, \cdots, c_n, z)$ noch in $|z|<2$ regulär (meromorph) sein. Dann ist $f(w_1, \cdots, w_n, z)$ in

$$\{|w_1|<1, \cdots, |w_n|<1, |z|<2\}$$

regulär (meromorph)." Der Satz war bisher nur für reguläre Funktionen bewiesen. Verf. gibt nun einen neuen Beweis an, der sich dann auch auf meromorphe Funktionen anwenden lässt. *P. Thullen* (Panamá).

Fuks, B. A. Natural boundaries of analytic functions of complex variables. Uspehi Matem. Nauk (N.S.) 5, no. 4(38), 75–120 (1950). (Russian)

This paper is a review, accompanied by proofs, of well-known results about natural boundaries of analytic functions of several complex variables, results initiated by Hartogs [Math. Ann. 62, 1–88 (1906)] and continued by Thullen [ibid. 106, 64–76 (1932)], Cartan and Thullen [ibid., 617–647 (1932)], and Oka [Jap. J. Math. 17, 517–521, 523–531 (1941); Tôhoku Math. J. 49, 15–52 (1942); these Rev. 3, 85; 7, 290]. The connections with the kernel function of Bergman are not discussed. *A. Zygmund.*

Tsuji, Masatsugu. On removable singularities of an analytic function of several complex variables. J. Math. Soc. Japan 1, 282–286 (1950).

Let E be a closed set in n-dimensional space, and D a region containing E. Suppose $u(P)=u(x_1, \cdots, x_n)$ is a bounded harmonic function in $D-E$, or for the case when n is even, $n=2k$, suppose $f(z_1, \cdots, z_k)$ is a bounded regular function in $D-E$. The author obtains conditions on the set E which enable him to conclude that u must be harmonic on E, or that f must be regular on E. One of his theorems states that if the set E is such that there exists a positive harmonic function $v(P)$ in $D-E$, such that $\lim v(P)=+\infty$ when P tends to any point of E, then any function $u(P)$ which is bounded and harmonic in $D-E$ will be harmonic on E. As an application of this result he obtains the following classical result [W. F. Osgood, Lehrbuch der Funktionentheorie, vol. II, part I, 2d ed., Leipzig, 1929]: Let $g(z_1, \cdots, z_k)$ be regular in a domain D and let E be the manifold defined by $g(z_1, \cdots, z_k)=0$. Then any function $f(z_1, \cdots, z_k)$ which is bounded and regular in $D-E$ will be regular on E. Several of the results are closely related to results in removable singularities presented by Bochner and the reviewer [Several Complex Variables, Princeton University Press, 1948; these Rev. 10, 366]. *W. T. Martin* (Cambridge, Mass.).

Poor, Vincent C. On the two-dimensional derivative of a complex function. Proc. Amer. Math. Soc. 1, 687–693 (1950).

The areolar derivative of Cioranescu of a polygenic function $w=f(z)$ over a certain domain D of the plane of complex numbers $z=x+iy$ is defined as the limit of a certain difference quotient. This limit is $w_{zz}-w_{\bar{z}\bar{z}}e^{-4i\theta}$, where the subscripts denote the applications of the mean and phase derivatives $\partial/\partial z=\frac{1}{2}(\partial/\partial x-i\partial/\partial y)$, and $\partial/\partial \bar{z}=\frac{1}{2}(\partial/\partial x+i\partial/\partial y)$, used frequently in the papers of Kasner and De Cicco [e.g., Univ. Nac. Tucumán. Revista A. 4, 7–45 (1944)] and De Cicco [Scripta Math. 11, 51–56 (1945); these Rev. 7, 59]. This Cioranescu derivative can be obtained by first differentiating along a direction of inclination θ, and then differentiating the result along an orthogonal rectilinear direction. The monogenic functions of Cioranescu obey $w_{\bar{z}\bar{z}}=0$. Thus the class of Cioranescu monogenic functions is characterized by the fact that the phase derivative $w_{\bar{z}}$, which is Pompieu's areolar derivative, is a monogenic function of z in the ordinary sense. Consequently, this class of Cioranescu monogenic functions is $w=\bar{z}g(z)+h(z)$, where g and h are monogenic functions of z. By differentiating with respect to $\bar{z}=x-iy$ instead of $z=x+iy$, the author considers dual Cioranescu monogenic functions. The various theorems obtained are those stated above in which z is replaced by $\bar{z}$, and $\bar{z}$ by z. Evidently the only polygenic functions of both types are $az\bar{z}+bz+c\bar{z}+d$, where a, b, c, d are complex constants. The author also considers the integral representations of polygenic functions, and shows how the preceding work is related to that of R. N. Haskell [Bull. Amer. Soc. 52, 332–337 (1946); these Rev. 7, 381] and M. O. Reade [ibid. 53, 98–103 (1947); these Rev. 8, 453]. *J. De Cicco.*

Theory of Series

Goodstein, R. L. On the multiplication of series. Math. Gaz. 34, 16–18 (1950).

Proofs of standard theorems on Cauchy products of series, based on a familiar theorem on transformations of sequences. *R. P. Agnew* (Ithaca, N. Y.).

Schur, Zvi. Oscillations of sequences in linear transformations. Riveon Lematematika 4, 29–34 (1950). (Hebrew. English summary)

The author gives six conditions, involving a normal matrix a_{mn} and a general matrix b_{mn}, necessary and sufficient to ensure that the oscillation of the sequence $\sum b_{mn}x_n$ does not

exceed the oscillation of the sequence $\sum a_{mn}x_n$ whenever x_n is a sequence for which the sequence $\sum a_{mn}x_n$ is bounded.
R. P. Agnew (Ithaca, N. Y.).

Robinson, A. On functional transformations and summability. Proc. London Math. Soc. (2) 52, 132–160 (1950).

The familiar Silverman-Toeplitz theorem, giving conditions on a matrix $A_{k,n}$ of complex constants necessary and sufficient to ensure that the sequence $t_k = \sum A_{k,n}s_n$ exists and converges to s whenever s_n is a sequence of complex numbers converging to s, is generalized. The s_n are now assumed to be elements of a Banach space S, and the $A_{k,n}$ are assumed to be linear bounded operators carrying elements of S into the same S. After giving an appropriate definition of the norm of a sequence of linear operators, the author is able to obtain conditions very similar to the Silverman-Toeplitz conditions. The familiar theorem on regularity of series-to-sequence transformations is likewise generalized. The assumption that the $A_{k,n}$ are bounded is then removed, and similar theorems are obtained. Some extensions and applications are suggested. *R. P. Agnew* (Ithaca, N. Y.).

Obrechkoff, Nikola. Sur quelques questions de la sommation des séries. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 37, 363–498 (1941). (Bulgarian. French summary)

"Dans ce travail nous considérons quelques problèmes de la théorie des séries divergentes. La série (1) $a_0+a_1+a_2+\cdots$ est dite sommable par la méthode de Poisson et Abel, bref sommable A, avec la somme s, si $f(x)=\sum_{n=0}^{\infty}a_nx^n$, $|x|<1$, est convergente et tend vers s lorsque $x\rightarrow 1-0$. Nous donnons une sommation plus générale, en combinant le procédé A et le procédé C de Cesàro. Nous disons que la série (1) est sommable (A, k) avec la somme s, si l'intégrale $f_k(x)=k(1-x)^{-k}\int_x^1(t-x)^{k-1}f(t)dt$ existe pour $0<x<1$ et tend vers s, lorsque $x\rightarrow 1-0$, en restant réel. Il est presque évident que chaque série sommable A est aussi sommable (A, k), avec la même somme, si $k>0$. Nous avons: si (1) est sommable (A, k), elle est aussi sommable (A, k_1) avec la même somme, si $k_1>k$. Nous démontrons encore un théorème qui généralise celui de Tauber: si la série (1) est sommable (A, k) et si $na_n=o(1)$ pour $n\rightarrow\infty$, la série (1) est convergente." *From the author's summary.*

Obrechkoff, N. Sur la sommation des séries par les moyennes typiques. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 41, 103–141 (1945). (Bulgarian. French summary)

Delange, Hubert. Sur les théorèmes inverses des procédés de sommation des séries divergentes. I. Ann. Sci. École Norm. Sup. (3) 67, 99–160 (1950).

Delange, Hubert. Sur les théorèmes inverses des procédés de sommation des séries divergentes. II. Ann. Sci. École Norm. Sup. (3) 67, 199–242 (1950).

These papers give detailed expositions of results of the author, concerning transforms of series, sequences, and functions satisfying Tauberian conditions, which were summarized earlier [C. R. Acad. Sci. Paris 225, 28–31 (1947); 226, 1787–1790 (1948); these Rev. 9, 28; 10, 32]. In addition to the several papers cited by the author, others of the author [ibid. 224, 436–438, 1802–1804 (1947); 225, 483–485 (1947); these Rev. 8, 457; 9, 27, 140] and Rajagopal [Proc. Indian Acad. Sci. Sect. A. 28, 537–544 (1948); 31, 60–61 (1950); these Rev. 10, 447; 12, 21] are relevant.
R. P. Agnew (Ithaca, N. Y.).

Lauwerier, H. A. Some Tauberian theorems. Mathematica, Zutphen B. 13, 62–75 (1946). (Dutch)

Exposition of Tauberian theorems for the Abel, Dirichlet series, and Laplace transformations. *R. P. Agnew.*

***Vernotte, Pierre. Abrégé de la théorie générale des séries divergentes dite théorie des séries définissables.** Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 36, i+41 pp. (1950).

Vernotte, Pierre. Sur l'interdépendance des termes de rang pair et des termes de rang impair d'une même suite. Application à la sommation des séries divergentes. C. R. Acad. Sci. Paris 231, 104–106 (1950).

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Remarks, not based on precise definitions, similar to those in the author's book [Théorie et pratique des séries divergentes . . . Publ. Sci. Tech. Ministère de l'Air, Paris, no. 207 (1947); these Rev. 11, 97]. *R. P. Agnew.*

Bosanquet, L. S. An extension of a theorem of Andersen. J. London Math. Soc. 25, 72–80 (1950).

Let ρ, k, and λ be real constants such that $\rho<0$, $k>-1$, $\rho+\lambda<0$, and $k+\lambda>-1$; and let $S^{-\lambda}(u_n)=\sum_{\nu=0}^{n}\binom{n-\nu-\lambda-1}{n-\nu}u_\nu$. The author shows that if $\sum n^\rho u_n$ is summable (C, k), then $\sum n^{\rho+\lambda}S^{-\lambda}(u_n)$ is summable $(C, k+\lambda)$, and conversely. He proves a corresponding theorem for nonnegative values of ρ, and he states that the theorems remain true when Cesàro summability is replaced by absolute Cesàro summability. Space does not permit a review of the motivation presented by the author. *G. Piranian* (Ann Arbor, Mich.).

Borwein, D. On the Cesàro summability of integrals. J. London Math. Soc. 25, 289–302 (1950).

The following and related theorems are proved. When $\rho<0$ and $0<a\leqq\lambda$, a necessary and sufficient condition that the infinite integral $\int_1^\infty t^\rho g(t)dt$ be evaluable to L by the Cesàro method (C, λ) (or by the absolute method $|C, \lambda|$) is that the infinite integral $\int_1^\infty t^{\rho-a}[\Gamma(a)]^{-1}\int_1^t(t-u)^{a-1}g(u)du\,dt$ be similarly evaluable to $L\Gamma(-\rho)/\Gamma(a-\rho)$. Analogous results for series have been established by Bosanquet [see the preceding review].
R. P. Agnew (Ithaca, N. Y.).

Borwein, D. A summability factor theorem. J. London Math. Soc. 25, 302–315 (1950).

The following theorem on Cesàro transforms of integrals of product integrands is proved. Let $\lambda\geqq 0$. If $\varphi(t)$ has a derivative of order λ which is absolutely continuous over $t\geqq 1$, and is such that $\int_0^\infty f(t)\varphi(t)dt$ is bounded (or evaluable) C_μ for some $\mu>0$ whenever $\int_0^\infty f(t)dt$ is evaluable (or bounded) C_λ, then (i) there is an absolutely continuous function $\psi(t)$ such that $\psi(t)=\varphi(t)$ almost everywhere and $\psi(t)$ is convergent (or convergent to 0) as $t\rightarrow\infty$ and

$$\text{(ii)}\qquad \int_1^\infty t^\lambda|\varphi^{(\lambda+1)}(t)|dt<\infty.$$

For results and references on corresponding problems involving product series, see Bosanquet [J. London Math. Soc. 17, 166–173 (1942); these Rev. 4, 194].
R. P. Agnew (Ithaca, N. Y.).

Cossar, J. A note on Cesàro summability of infinite integrals. J. London Math. Soc. 25, 284–289 (1950).

It is shown that if $r>0$, if k is the smallest integer not less than r, and if $\varphi(x)$ is a function not identically zero for large x which has a kth derivative absolutely continuous over each finite interval $(0, a)$, then there is a function $f(x)$ such that

the integral $\int_0^\infty f(x)dx$ is evaluable by the Cesàro method C_r, while the integral $\int_0^\infty f(x)\varphi(x)dx$ is nonsummable C_s for each s less than r. Some cases, obtained by specializing $\varphi(x)$, r, and s, were previously known. *R. P. Agnew.*

Avakumović, Vojislav G. Bemerkung über einen Satz des Herrn T. Carleman. Math. Z. **53**, 53–58 (1950).

Let $f(x)=x\int_0^\infty (u+x)^{-1}dS(u)=O(e^{-\sqrt{x}})$ as $x\to\infty$. Then under the Tauberian condition $u^{\frac12}\{S(v)-S(u)\}>-m$ for $u\leqq v\leqq u+u^{\frac12}$, it follows that $u^{\frac12}S(u)=O(1)$ as $u\to\infty$. The proof proceeds by showing by elementary means that $v^S(v^2)-u^S(u^2)>-m$ (with a different m) for $u\leqq v\leqq u+1$ and that $f(x)=O(1)$ as $x\to 0$; and then multiplying $f(x)$ by $(sx)^{-1}\sin(sx^{\frac12})$, integrating over $(0,\infty)$, and reversing the order of integration. The result is a function of the form $s^{-1}\int_0^\infty e^{-s\sqrt{u}}dS(u)$ which is regular to the right of the imaginary axis and at 0. A further transformation leads to a Laplace transform to which a known Tauberian theorem can be applied. It is indicated how a sharper result can be obtained from a stronger hypothesis and how the following analogous result can be proved: If $h(t)=\int_0^\infty e^{-tu}dS(u)=O(e^{-1/t})$ as $t\to 0$, then the same conclusion about $S(u)$ holds under the same Tauberian condition. *R. P. Boas, Jr.*

✱Perron, Oskar. Die Lehre von den Kettenbrüchen. 2d ed. Chelsea Publishing Co., New York, N. Y., 1950. xii+524 pp.

Photographic reproduction of a book published in 1929 by Teubner, Leipzig.

Fourier Series and Generalizations, Integral Transforms

Sunouchi, Gen-Ichirô. On the strong summability of Fourier series. Proc. Amer. Math. Soc. **1**, 526–533 (1950).

The author proves here three apparently scattered results which are pieced together by a certain similarity in the methods of proof. First, he gives a new proof of a known result of Marcinkiewicz and Zygmund [Duke Math. J. **4**, 473–485 (1938)]. Secondly, an earlier result of the author is completed [Proc. Imp. Acad. Tokyo **19**, 420–423 (1943); these Rev. **7**, 247]. Thirdly, a theorem which includes the following is proved. If $f(\theta)\in L^r$, $r>1$, and $\{p_n\}$ is an increasing sequence of positive integers such that $p_n/n=O(p_{n+1}-p_n)$, then $\sum_{n=1}^\infty |s_{p_n}-f|^m/n<\infty$, $m\geqq 1$, for almost all θ. The s stands for the partial sum of the Fourier series of f.
K. Chandrasekharan (Bombay).

Sunouchi, Gen-Ichirô. On double Fourier series. Proc. Amer. Math. Soc. **1**, 522–525 (1950).

Following the methods of Marcinkiewicz, the author proves some inequalities between functions of two variables and the partial sums of their Fourier series. Here is a sample: For $\gamma>1$, we have

$$\int_0^{2\pi}\int_0^{2\pi}|\max_{m,n} s_{2^m,2^n}(x,y)|^\gamma dx\,dy\leqq D_\gamma\int_0^{2\pi}\int_0^{2\pi}|f(x,y)|^\gamma dx\,dy.$$

K. Chandrasekharan (Bombay).

Izumi, Shin-ichi. Notes on Fourier analysis. VIII. Local properties of Fourier series. Tôhoku Math. J. (2) **1**, 136–143 (1950).

The author discusses the absolute summability of a Fourier series by the logarithmic mean $(\log n)^{-1}\sum_{\nu=1}^n s_\nu/\nu$, which he calls summability $|R, \log n, 1|$. He first proves that summability $|R, \log n, 1|$ at a point is not a local property, i.e., that a function may be zero in the neighborhood of a point without its Fourier series being summable $|R, \log n, 1|$ there [the corresponding property for $|C, 1|$ summability was given by the reviewer and Kestelman, Proc. London Math. Soc. (2) **45**, 88–97 (1939)]. On the other hand, he proves that summability $|R, \log n, 1|$ throughout an interval is a local property in the sense of Wiener; that is, if there is a set of functions each agreeing with $f(x)$ in one of a set of overlapping intervals covering $(-\pi, \pi)$ and each having its Fourier series summable $|R, \log n, 1|$, then the Fourier series of $f(x)$ is summable $|R, \log n, 1|$ [the corresponding property for absolute convergence was given by Wiener, Ann. of Math. (2) **33**, 1–100 (1932), lemma IIb; The Fourier Integral . . . , Cambridge University Press, 1933, lemma 6_{15}]. Summability $|R, \lambda_n, 1|$ was defined by Obrechkoff [C. R. Acad. Sci. Paris **186**, 215–217 (1928)] as the bounded variation of the Riesz mean $a(\omega)=\sum_{\lambda_n<\omega}(1-\lambda_n/\omega)a_n$. Since $a(\omega)$ is monotonic in $(\lambda_n, \lambda_{n+1})$ this property is equivalent to the bounded variation of the sequence $a(\lambda_{n+1})$. According to this definition, the author's method would be summability $|R, L_n, 1|$, where $L_n=1+\frac12+\cdots+1/n$. But it may be shown that summability $|R, L_n, 1|$ is equivalent to summability $|R, \log n, 1|$ (in Obrechkoff's sense) [the corresponding equivalence for nonabsolute summability was stated by M. Riesz, C. R. Acad. Sci. Paris **149**, 18–21 (1909); see Hardy, Divergent Series, Oxford, 1949, theorem 37; these Rev. **11**, 25]. This justifies the author's terminology, and also shows that his first result is equivalent to one obtained independently by R. Mohanty [J. London Math. Soc. **25**, 67–72 (1950); these Rev. **11**, 592]. *L. S. Bosanquet* (London).

Matsuyama, Noboru. On the jump of a function and its Fourier series. Notes on Fourier analysis. XXXIII. J. Math. Soc. Japan **1**, 212–218 (1950).

Let $f(x)\sim\sum_1^\infty b_n \sin nx$. If $f(x)$ has a jump of $2S$ at $x=0$ in the sense that the Riesz logarithmic average of order $\alpha>0$ of $f(x)$ as $x\to 0^+$ has a limit S, then the $(R, \log n, \alpha+1+\delta)$ means of the sequence $\{nb_n\}$ have a limit $2S/\pi$. Conversely, if $\lim \{nb_n\}=2S/\pi$ $(R, \log n, \alpha)$, $\alpha\geqq 1$, then $\lim_{x\to 0} f(x)=S$ $(R, \log n, \alpha+1+\delta)$. *P. Civin* (Eugene, Ore.).

✱Men'šov, D. E. On convergence in measure of trigonometric series. Trudy Mat. Inst. Steklov. **32**, 99 pp. (1950). (Russian)

This paper gives complete proofs of results announced earlier without proof [Doklady Akad. Nauk SSSR (N.S.) **59**, 849–852 (1948); these Rev. **9**, 426]. *A. Zygmund.*

Men'šov, D. On the convergence of trigonometric series. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 170–184 (1950). (Russian)

This paper is a review, with some proofs, of the work done by the author within the last ten years in the field of convergence of trigonometric (not necessarily Fourier) series. For the statement of the results see Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 667–692 (1941); **15(57)**, 385–432 (1944); **20(62)**, 197–238 (1947); C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 245–246 (1941); **41**, 51–53 (1943); **49**, 79–82 (1945); Doklady Akad. Nauk SSSR (N.S.) **59**, 849–852 (1948); these Rev. **3**, 106; **6**, 264; **8**, 577; **3**, 106; **6**, 47; **7**, 435; **9**, 426. *A. Zygmund* (Chicago, Ill.).

Men'šov, D. On the limits of indeterminateness of trigonometric series. Doklady Akad. Nauk SSSR (N.S.) **74**, 181–184 (1950). (Russian)

An earlier result of the author [Rec. Math. [Mat. Sbornik] N.S. 9(51), 667–692 (1941); these Rev. **3**, 106] asserts that, given any measurable and almost everywhere finite function $f(x)$ of period 2π, there is a trigonometric series (*) $\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx)$ with coefficients tending to zero, and converging to $f(x)$ almost everywhere. This series is not unique and is not necessarily the Fourier series of $f(x)$ if $f(x)$ is integrable. The paper under review gives, without proof, the following result, which generalizes the one given above. Let $F(x)$ and $G(x)$ be two functions measurable on $(-\pi, \pi)$ and satisfying the inequality $F(x)\leqq G(x)$ almost everywhere. Then there is a trigonometric series (*), with coefficients tending to 0 and with partial sums $s_n(x)$ satisfying $\liminf_{n\to\infty} s_n(x)=F(x)$, $\limsup_{n\to\infty} s_n(x)=G(x)$ almost everywhere. The functions $F(x)$ and $G(x)$ are supposed to be either both finite almost everywhere or, if one of them is $\pm\infty$ in a set E of positive measure, then the other is $\mp\infty$ almost everywhere in E. Another result of a similar type stated by the author is as follows. For any function $\varphi(x)\geqq 0$ defined on $(-\pi, \pi)$ and measurable, there is a function $f(x)$, integrable L and such that if (*) is the Fourier series of f, then $\liminf_{n\to\infty} s_n(x)=f(x)-\varphi(x)$, $\limsup_{n\to\infty} s_n(x)=f(x)+\varphi(x)$ almost everywhere.

A. Zygmund (Chicago, Ill.).

Ingham, A. E. A further note on trigonometrical inequalities. Proc. Cambridge Philos. Soc. **46**, 535–537 (1950).

The author proves that if $f(t)=\sum_N^{N'}a_ne^{-\lambda_n ti}$ with real λ_n satisfying $\lambda_n-\lambda_{n-1}\geqq\gamma>0$ $(N<n\leqq N')$, and $\gamma T=\pi$, then

$$|a_n|\leqq T^{-1}\int_{-T}^{T}|f(t)|\,dt, \qquad N\leqq n\leqq N';$$

the right hand side cannot be multiplied by a factor smaller than 1. This improves an earlier result of the author [Math. Z. 41, 367–379 (1936)] and implies improvements in applications which were made in that paper.

R. P. Boas, Jr. (Evanston, Ill.).

***Bochner, S. Localization of best approximation.** Contributions to Fourier Analysis, pp. 3–23. Annals of Mathematics Studies, no. 25. Princeton University Press, Princeton, N. J., 1950. **$3.00.**

The purpose of this paper is to prove a new type of theorem on the approximation of functions by polynomials. The motivation is supplied by the classical localization theorem on Fourier trigonometric series which states that if a periodic, Lebesgue integrable function vanishes in a neighborhood of a point x_0, then the sequence of partial sums of its Fourier series converges to zero at that point. The author considers the more general situation where such a function remains constant, not necessarily zero, in intervals, and associates with it a sequence of exponential polynomials which, in general, approximate to it like Fejér polynomials, and at points of constancy approximate to it with an error $O(e^{-n\epsilon(n)})$ uniformly in the interval of constancy. The principal result is as follows. Given a sequence of positive numbers $\epsilon(n)$ converging to zero as $n\to\infty$, no matter how slowly, the author assigns to any periodic function $f(x)\sim\sum a_ne^{inx}$, belonging to the class L, a sequence of exponential polynomials $\{s_n(f; x)\}$ (the subscript n always denoting a bound for the exponents) such that (i) the sequence has all the features of a Fejér sequence [it approximates to $f(x)$ in L-norm, and also in L_p or C-norm if $f(x)$ belongs to L_p or C, $s_n(x)\to f(x)$ at points of continuity or simple discontinuity and for a bounded function $f(x)$ one has $\inf_x f(x)\leqq s_n(x)\leqq\sup_x f(x)$], (ii) if in an interval $a<x<b$ the function $f(x)=c$, a constant, then $s_n(x)-c=O(e^{-n\epsilon(n)})$ uniformly in every closed sub-interval $a+\delta\leqq x\leqq b-\delta$. The proof of the theorem rests on a combination of the technique of de La Vallée Poussin in the theory of approximation of periodic functions by the Fejér means of their Fourier series, with the author's own technique of approximation in his theory of spherical means. The latter makes it possible to uphold his result not only for periodic but for almost periodic functions. A similar result also holds for approximation of nonperiodic functions by ordinary polynomials, with the difference that, whereas in the periodic case the coefficients of the approximating trigonometric sums will be bounded for every function, in the nonperiodic case the maximal coefficient of the nth polynomial will tend to $+\infty$ with n. The author shows how all the results go through in the case of several variables. It is of interest to note that the degree of approximation in the theorem cited above cannot be improved upon. The author establishes this fact by the following theorem: If for a sequence of exponential polynomials $\{\sigma_n(x)\}$ one has $|\sigma_n(x)|\leqq 1$ and if for three constants $\alpha>0$, $\epsilon_0>0$, $c_1>0$, one has $|\sigma_n(x)|\leqq c_1e^{-n\epsilon_0}$ for x in $-\alpha\leqq x\leqq\alpha$, then the sequence $\{\sigma_n(x)\}$ converges to zero uniformly in all of $-\pi\leqq x\leqq\pi$. A generalization of the theorem is indicated where piecewise constancy is replaced by piecewise analyticity.

K. Chandrasekharan.

Calderón, A. P. On theorems of M. Riesz and Zygmund. Proc. Amer. Math. Soc. **1**, 533–535 (1950).

The theorems in question are: (a) The conjugate of the Fourier series of a function $f(x)$ of L^p, $p>1$, is the Fourier series of a function $\tilde f(x)$ of the same class, and $\int_0^{2\pi}|\tilde f(x)|^p dx\leqq A_p\int_0^{2\pi}|f(x)|^p dx$; (b) a similar result with $f(x)\log^+|f(x)|$ and $\tilde f(x)$ both of class L [see Zygmund, Trigonometrical Series, Warsaw-Lwow, 1935, pp. 147–151]. New proofs are given by the author.

L. S. Bosanquet.

***Calderón, A. P., and Zygmund, A. On the theorem of Hausdorff-Young and its extensions.** Contributions to Fourier Analysis, pp. 166–188. Annals of Mathematics Studies, no. 25. Princeton University Press, Princeton, N. J., 1950. **$3.00.**

This paper consists of an introduction and four parts. The first part gives a proof of the Hausdorff-Young-F. Riesz theorem without passing through the apparatus of M. Riesz's theorem on bilinear forms: the only tool used is a simple application of the maximum principle in the Phragmén-Lindelöf form for a strip. The second part deals with the extreme cases in which equality occurs. In the third part, the method already applied in part I leads directly to the proof of M. Riesz's convexity theorem for linear operations on functions of the classes L^p. The fourth part contains an extension of M. Riesz's convexity theorem on linear operations applied to functions of the Hardy classes H^p, of the same kind as given in a paper by the reviewer and Zygmund [Proc. Nat. Acad. Sci. U. S. A. **34**, 443–447 (1948); these Rev. **10**, 247].

R. Salem (Cambridge, Mass.).

Obrechkoff, N. Sur les systèmes biorthogonaux de fonctions analytiques. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **38**, 49–86 (1942). (Bulgarian. French summary)

"Dans ce travail nous nous occupons du problème général du developpement suivant un système de fonctions ana-

lytiques $\varphi_n(z)$ et suivant des systèmes biorthogonaux des fonctions analytiques." *From the author's summary.*

Castoldi, L. La funzione di Dirac e un criterio di completezza per sistemi di funzioni ortonormali. Nuovo Cimento (9) **7**, Supplemento no. 1, 54–66 (1950).

This paper is concerned with the point-wise convergence of a mixed series and integral expansion of a function in terms of a mixed discrete-continuous orthonormal system. Practically all results obtained are false; for example, it is found that (1) the Fourier series of every continuous periodic function converges to the function at every point; (2) if in a Sturm-Liouville boundary value problem, every function of some class C^n $(n>0)$ is, at every point, equal to the sum of its eigenfunction expansion, then the same is true for any continuous function. *J. Korevaar* (Lafayette, Ind.).

Erdélyi, A. The inversion of the Laplace transformation. Math. Mag. **24**, 1–6 (1950).

The author points out that the various known inversion formulas for the Laplace transform may all be derived from the following principle: "If there is a singular integral whose kernel $N(u, t; k)$ can be interpreted as the result of a linear operation $L_{k,t}$ on e^{-tu}, then $L_{k,t}$ is an inversion operator for the Laplace transform." This idea may also be applied to the systematic invention of new inversion formulas. Many illustrations are given. *I. I. Hirschman, Jr.*

Bose, S. K. Certain properties of the generalized Laplace transform. J. Indian Math. Soc. (N.S.) **14**, 29–34 (1950).

In this sequel to earlier papers [Bull. Calcutta Math. Soc. **41**, 9–27, 59–67, 68–76 (1949); **42**, 43–48 (1950); these Rev. **11**, 28, 173; **12**, 95] the author obtains two theorems from generating functions of Laguerre and Hermite polynomials, respectively, by multiplying by an arbitrary function and integrating. *A. Erdélyi* (Pasadena, Calif.).

Richardson, Albert S., Jr. The remainder theorem and its application to operational calculus techniques. Proc. I.R.E. **38**, 1336–1339 (1950).

Special Functions

✱Low, A. R. Normal Elliptic Functions. A Normalized Form of Weierstrass's Elliptic Functions. University of Toronto Press, Toronto, Ont., 1950. ii+30 pp. $1.25.

The author introduces a normalised form for the Weierstrass elliptic integral $U=\frac{1}{2}\int_z^\infty\{(z-e_1)(z-e_2)(z-e_3)\}^{-1/2}dz$ by the substitutions $v=(z-e_3)/(e_1-e_3)$, $m=(e_2-e_3)/(e_1-e_3)$. The result is (1) $u=U(e_1-e_3)^{-1/2}=\frac{1}{2}\int_v^\infty\{v(v-m)(v-1)\}^{-1/2}dv$. Four types of elliptic integrals are defined corresponding to (1), one in each of the ranges $(\infty\rightarrow 1)$, $(1\rightarrow m)$, $(m\rightarrow 0)$, and $(0\rightarrow -\infty)$. By inverting, four elliptic functions p_k $(k=1,2,3,4)$ are found which satisfy $p_1(u,m)\cdot p_3(u,m)=m$, $p_2(u, 1-m)+p_3(u, m)=1$, $p_1(u, m)=ns^2(u, k)$, and $p_1(u, m)-p_4(u, 1-m)=1$. From these relations values of p_1, p_2, p_3, and p_4 are computed from Milne-Thomson's tables of Jacobi's elliptic functions for values of $u_k\omega^{-1}$ from 0 to 1, and of m from 0 to 1, by steps of 0.1. *S. C. van Veen.*

Kaplan, E. L. Multiple elliptic integrals. J. Math. Physics **29**, 69–75 (1950).

The author gives a set of formulas concerning multiple elliptic integrals. These formulas contain a great number of definite and indefinite integrals which are reducible to elliptic integrals of the first and of the second kind. Some numerical results are included. *M. J. O. Strutt.*

Šura-Bura, M. R. Evaluation of an integral containing a product of Bessel functions. Doklady Akad. Nauk SSSR (N.S.) **73**, 901–903 (1950). (Russian)

It is known that (*) $\int_0^\infty e^{-at}J_0(bt)J_1(ct)dt$ can be expressed as a finite integral, and that the integral in question is an elliptic integral. The author carries out the reduction to a complete elliptic integral of the third kind and thereby proves that (*) is equal to $c^{-1}(1-\Lambda_0(\alpha, \beta))$, where Λ_0 is a function for which numerical tables are available [Heuman, J. Math. Phys. **20**, 336 (1941); these Rev. **3**, 152].

A. Erdélyi (Pasadena, Calif.).

Sips, Robert. Convergence des séries représentant les fonctions de Mathieu et les fonctions d'onde sphéroïdales. II. Bull. Soc. Roy. Sci. Liège **19**, 55–71 (1950).

As a sequel to his previous paper on this subject [same Bull. **18**, 498–515 (1949); these Rev. **11**, 663] the author considers the subsequent approximations and the solutions of the Mathieu equation by a Fourier series. The coefficients of these series are determined by a set of recurrent equations. He proves that they satisfy certain limiting equations so as to correspond to convergent series. In the case of spheroidal functions the series are formed by Legendre functions. The procedure here is similar to the one above. Some numerical results are included. *M. J. O. Strutt.*

Sips, Robert. Convergence des séries représentant les fonctions de Mathieu et les fonctions d'onde sphéroïdales. III. Bull. Soc. Roy. Sci. Liège **19**, 107–118 (1950).

[Cf. the preceding review.] The author starts to derive approximate expressions for the separation constant of the differential equations of the above functions, which are being developed in series of associated Legendre functions. He then proceeds to extend the domain of convergence of the series under consideration. In these calculations he utilises the approximate expressions mentioned above.

M. J. O. Strutt (Zurich).

Wintner, Aurel. On the Whittaker functions $W_{km}(x)$. J. London Math. Soc. **25**, 351–353 (1950).

The author derives from a general theorem [Amer. J. Math. **69**, 87–98 (1947); these Rev. **8**, 381] the apparently new fact that the functions of the title are completely monotonic on $(0, \infty)$ if $k\leqq 0$, $m\geqq\frac{1}{2}$, $(k, m)\neq(0, \frac{1}{2})$.

R. P. Boas, Jr. (Evanston, Ill.).

Tricomi, F. G. Über die Abzählung der Nullstellen der konfluenten hypergeometrischen Funktionen. Math. Z. **52**, 669–675 (1950).

The real zeros of Whittaker's function $M_{k,m}(x)$ have been investigated by Kienast [Denkschriften der Schweizerischen Naturforschenden Gesellschaft **57**, 247–325 (1921)]; Tsvetkoff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 10–12 (1941); (N.S.) **33**, 290–291 (1941); these Rev. **3**, 237; **5**, 120] has rediscovered Kienast's results and in addition stated some results on the zeros of the second Whittaker function $W_{k,m}(x)$. In the present paper the author gives a simple and elegant formulation (by means of a diagram) of Kienast's results, and a similar formulation, and also proof, of Tsvetkoff's statements. *A. Erdélyi* (Pasadena, Calif.).

Sears, D. B. On the transformation theory of hypergeometric functions. Proc. London Math. Soc. (2) **52**, 14–35 (1950).

It has been established by various authors that identities in power series can be generalized frequently by replacing the power terms by suitable arbitrary sequences of terms. In the present case the substitution involves the displacement and difference operators in addition to the sequence of terms. Several broadly inclusive identities are established under clearly stated conditions. These are shown to generalize and include a variety of established hypergeometric series identities. Typical, but among the simpler results obtained, is the identity

$$ {}_1F_0(a; xE)\theta_0=(1-x)^{-a}{}_1F_0(a; x\Delta/(1-x))\theta_0, $$

where E is the displacement and Δ the difference operator applied to θ_0 as a generator for the sequence, θ_r.

N. A. Hall (Minneapolis, Minn.).

Cherry, T. M. Asymptotic expansions for the hypergeometric functions occurring in gas-flow theory. Proc. Roy. Soc. London. Ser. A. **202**, 507–522 (1950).

In applying the hypergeometric functions arising in the solution of gas flow problems by the hodograph method there is occasion to refer to expressions for their functions of large order asymptotic in the order ν. Previous results, developed herein more concisely, involve exponential or circular functions and give unsatisfactory approximations near the sonic point. New formulae involving Bessel functions are established to terms in ν^{-4}, giving uniform approximation over the range of significant interest. The coefficients in the asymptotic expression are tabulated to give seven figure accuracy for $|\nu|\geqq 10$ for the range of function argument $\tau=0(.02).50$ with $\gamma=1.4$.

N. A. Hall.

Erdélyi, A. Hypergeometric functions of two variables. Acta Math. **83**, 131–164 (1950).

Hypergeometric functions of two variables are solutions associated with the system of partial differential equations

$$ (1)\quad \begin{aligned} &x(1-x)r+y(1-x)s+\{\gamma-(\alpha+\beta+1)x\}p-\beta yq-\alpha\beta z=0,\\ &y(1-y)t+x(1-y)s+\{\gamma-(\alpha+\beta'+1)y\}q-\beta'xq-\alpha\beta'z=0. \end{aligned} $$

Before 1926 ten different solutions of (1) had been derived and these solutions are represented by sixty convergent series of the form

$$ x^\lambda(1-x)^\mu y^{\lambda'}(1-y)^{\mu'}(x-y)^\sigma\sum\sum\frac{(\lambda)_{m+n}(\mu)_m(\mu')_n}{(\nu)_{m+n}m!n!}t^m t'^n, $$

where $|t|<1$, $|t'|<1$, λ, μ, μ', ν, χ, χ', ρ, ρ', σ depend on α, β, β', γ, and t, t' are rational functions of x and y [cf. Appel and Kampé de Fériet, Fonctions hypergéométriques et hypersphériques . . . , Gauthier-Villars, Paris, 1926]. The author has undertaken to obtain all significant solutions of (1) by means of contour integration (in particular, along trefoil loops). The transformation theory leads to the general theorem: Any hypergeometric system of partial differential equations of the second order in two independent variables which has only three linearly independent integrals can be transformed into (1) or into a particular or a limiting case of (1). Among many other particular results obtained by the author we mention the integration of the system

$$ \begin{aligned} x(4x+1)r-(4x+2)ys+y^2t+\{(4\alpha'+6)x+1-\alpha\}p&\\ -2\alpha'yq+\alpha'(\alpha'+1)z&=0,\\ x^2r-x(4y+2)s+y(4y+1)t-2\alpha xp+\{(4\alpha+6)x+1-\alpha'\}q&\\ +\alpha(\alpha+1)z&=0. \end{aligned} $$

The integration of this system hitherto presented considerable difficulties owing to the presence of a cusp at $x=y=-\frac{1}{8}$ of its singular manifold $h=27x^2y^2-18xy-4x-4y-1=0$ and the presence of another apparent singular manifold.

S. C. van Veen (Delft).

Harmonic Functions, Potential Theory

Cartan, Henri, et Deny, Jacques. Le principe du maximum en théorie du potentiel et la notion de fonction surharmonique. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 81–100 (1950).

In classical potential theory in Euclidean n-space R^n, there are two fundamental problems, (i) the equilibrium problem, and (ii) the "sweeping out" of mass onto a given closed set. A solution to problem (i) is based upon the following "first principle of the maximum" due to Frostman [Thesis, University of Lund, 1935] and Maria [Proc. Nat. Acad. Sci. U. S. A. **20**, 485–489 (1934)]: If μ is a positive mass distribution with compact support, and if U^μ is its potential function, and if U^μ is dominated by the constant c on the support, then $U^\mu\leqq c$ throughout R^n. As M. Riesz has shown, by means of Kelvin transformations, the solution to (i), and a limiting process, one can obtain a solution to (ii). But the present authors wish to obtain a solution to (ii) independent of (i); to do this they isolate the following "second principle of the maximum." If U^μ and V^ν are the potentials due to positive mass distributions of finite energy, and if $U^\mu\leqq V^\nu$ except on a set of μ-measure zero, then $U^\mu\leqq V^\nu$ throughout R^n. The conjunction of the two principles above is called "the complete principle of the maximum."

The present authors (and the reviewer) use the definitions and notation associated with H. Cartan, Brelot, and L. Schwartz [see Deny, Acta Math. **82**, 107–183 (1950); these Rev. **12**, 98]. In the first section of the paper, the authors consider kernels $K(x)$ which are measures of positive type and which are "of slow growth": they then define several classes of nonnegative functions, in terms of which they give several necessary and sufficient conditions in order that the second principle of the maximum hold. They give next necessary and sufficient conditions that a given kernel admit affirmative replies to the problems (i) and (ii) above; one such condition is that the complete principle of the maximum prevail. In the second section of the paper, the authors consider "regular" kernels $K(x)$, by means of which they obtain results analogous to certain classical ones. For example, if μ is a measure of finite energy, then there exists a sequence of measures of compact support $\{\mu_n\}$, such that $\mu_n\uparrow\mu$; moreover, $U^{\mu_n}\uparrow U^\mu$ throughout R^n (Evans-Vasilesco, essentially). The authors obtain a necessary and sufficient condition that a regular kernel admit the complete principle of the maximum; they then associate a family $\mathfrak{F}$ of distributions $\{\sigma\}$, with each kernel admitting the complete principle of the maximum, to obtain the following analogue of a result due to F. Riesz [Acta Math. **54**, 321–360 (1930)]. If U^μ is the potential generated by the distribution μ, then $U^\mu(x)\geqq\int U^\mu(x+y)d\sigma(y)$, for all $\sigma\epsilon\mathfrak{F}$; i.e., potentials U^μ are "superharmonic." A converse is also obtained. The authors promise a sequel to the present paper in which the potentials of order α of M. Riesz and Frostman will be examined.

M. Reade (Ann Arbor, Mich.).

Leja, François. Une méthode d'approximation des fonctions réelles d'une variable complexe par des fonctions harmoniques. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 292–302 = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 276 (1950).

Let F denote the bounded frontier of an unbounded plane domain, and let the transfinite diameter $d(F)$ be positive. Now let $f(\zeta)$ denote a bounded real function of ζ on F, $m \leqq f(\zeta) \leqq M$, and let $\lambda \geqq 0$ be arbitrary, though fixed. Then the author uses Lagrange polynomials to prove the existence of a function $f(z, \lambda)$ with the properties: (i) $f(z, \lambda)$ is harmonic for $z \bar{\epsilon} F$, while $m\lambda \leqq f(\zeta, \lambda) \leqq \lambda f(\zeta)$ for $\zeta \epsilon F$; (ii) if $f(\zeta)$ is continuous on F, if $\lambda > 0$, then $\lambda^{-1} f(\zeta, \lambda) \rightrightarrows f(\zeta)$, as $\lambda \to 0$, for $\zeta \epsilon F$. Various other properties of the family $\{\lambda^{-1} f(z, \lambda)\}$ are studied, for $\lambda > 0$, as well as its use in solving the Dirichlet problem in the plane. *M. Reade* (Ann Arbor, Mich.).

Landkof, N. S. Approximation of continuous functions by harmonic functions. Mat. Sbornik N.S. 25(67), 95–106 (1949). (Russian)

L'auteur démontre et complète des résultats déjà annoncés [C. R. (Doklady) Acad. Sci. URSS (N.S.) 55, 7–8 (1947); ces Rev. 9, 29]. Il faut signaler que l'essentiel avait déjà été indiqué, avec une démonstration voisine, basée aussi sur le théorème des systèmes totaux de Banach, par J. Deny [Bull. Soc. Math. France 73, 71–73 (1945); ces Rev. 7, 205] qui est revenu sur le sujet avec plus de détails [voir l'analyse ci-dessous]. *M. Brelot* (Grenoble).

Deny, Jacques. Systèmes totaux de fonctions harmoniques. Ann. Inst. Fourier Grenoble 1 (1949), 103–113 (1950).

This paper deals with the approximation of functions which are continuous on the boundary of a compact set F in n-space by functions harmonic in a neighborhood of F. The author gives detailed proofs of results announced earlier [Bull. Soc. Math. France 73, 71–73 (1945); these Rev. 7, 205]. In addition, the author discusses a question of similar approximations when derivatives enter into consideration. Modern potential theoretic methods as well as Banach space methods are used throughout the paper. *M. Reade.*

Brelot, Marcel. Compléments à la théorie de J. Deny. Ann. Inst. Fourier Grenoble 1 (1949), 113–120 (1950).

The author extends the results of Deny [see the preceding review] to the compact space obtained by adjoining a point at infinity to n-space. *M. Reade* (Ann Arbor, Mich.).

Brelot, Marcel. Étude des fonctions sous-harmoniques au voisinage d'un point singulier. Ann. Inst. Fourier Grenoble 1 (1949), 121–156 (1950).

Let $S(P, r)$ denote the $(p-1)$-sphere with center P and radius r in Euclidean p-space, $p \geqq 2$; for summable f, let $M(f; P, r)$ denote the integral mean of f over $S(P, r)$. For a function u, subharmonic in a deleted neighborhood of P, the author has shown how bounds on $r^\lambda M(u; P, r)$ (as $r \to 0$, $\lambda > p-2$, if P is a finite point, and as $r \to \infty$, $\lambda > 0$, if P is the point at infinity) imply bounds on $|u|$, and he has shown how bounds on $r^\lambda M(u^+; P, r)$ imply bounds on u^+ (though not on $|u|$) [see Étude des fonctions sousharmoniques . . . , Actualités Sci. Ind., no. 139, Hermann, Paris, 1934; Ann. Sci. École Norm. Sup. (3) 61, 301–332 (1944); Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 22, 205–219 (1946); these Rev. 7, 204; 8, 581]. In the present paper, the author continues those earlier researches; en route, he obtains new results concerning the derivatives of Green's functions and a sharpening and extension of a recent result due to M. Heins [Ann. of Math. (2) 49, 200–213 (1948); these Rev. 9, 341].

The author's technique may be illustrated as follows. Let $p=3$, let u be subharmonic near the origin O, and let $rM(u^+; O, r)$ be bounded. Then $\lim M(u; O, r)$, $\lim rM(u; O, r)$, and $\lim rM(u^+; O, r)$, as $r \to 0$, all exist. If we assume also that $r^s M(u^+; O, r)$ is summable in r, for r near $r=0$, for some $s>0$, and if k is the integer $s-1 \leqq k < s$, and if $1/\overline{MP}$ is expanded in a series of Legendre polynomials $\sum_0^\infty P_n(\cos\gamma)\overline{OP}^n/\overline{OM}^{n+1}$, for fixed $M \neq O$, and if μ is the nonpositive mass distribution associated with u, then the function $v(M) = \int [1/\overline{MP} - \sum_0^k P_n(\cos\gamma)\overline{OP}^n/\overline{OM}^{n+1}] d\mu(P)$ has the following properties: (i) $v(M)$ is well defined; (ii) $\lim \overline{OM}^{s+1} \cdot v^+(M) = 0$, as $\overline{OM} \to 0$; and

$$\text{(iii)} \qquad \lim r^{s+1} M((u-v)^+; O, r) = 0,$$

as $r \to 0$. It follows that $\lim \overline{OM}^{s+1} \cdot u^+(M) = 0$, as $\overline{OM} \to 0$. This last result can be sharpened and deepened, as the author proceeds to do. The author considers allied problems, such as the convergence of the sequence of solutions of Dirichlet problems associated with a convergent nested sequence of domains. He also points out the immediate applications of his results and methods to problems in function theory, since $\log |f|$, for analytic $f(z)$, is (sub)-harmonic except for isolated points. *M. Reade.*

Aki, Kunio. A note on the generalized Laplacian operators. Kōdai Math. Sem. Rep. 1950, 11–12 (1950).

Let $u(P)$ be real and continuous in a simply-connected domain G in E^3, and let $M(u; P; r)$ and $m(u; P; r)$ denote the mean-values of $u(P)$ over the volume and surface, respectively, of the sphere with center P and radius r. Let $\bar{\Delta} u(P) = \limsup_{r\to 0} 15[m(u; P; r) - M(u; P; r)]/2r^2$ be a Blaschke-Privaloff operator. Then the author's main result is that a necessary and sufficient condition that $u(P)$ be subharmonic in G is that $\bar{\Delta} u(P) \geqq 0$ hold throughout G; this is analogous to a result stated without proof, for the plane, by the reviewer [Duke Math. J. 10, 531–536 (1943); these Rev. 5, 7]. The author's proof depends upon the unproved statement that if $u(P)$ has a local maximum at Q, then $\bar{\Delta} u(Q) \leqq 0$. *M. Reade* (Ann Arbor, Mich.).

✱Bochner, S. Dirichlet problem for domains bounded by spheres. Contributions to Fourier Analysis, pp. 24–45. Annals of Mathematics Studies, no. 25. Princeton University Press, Princeton, N. J., 1950. $3.00.

L'auteur reprend sans le savoir des recherches de Zaremba [J. Math. Pures Appl. (9) 6, 127–163 (1927)] et Nikodým [ibid. 12, 95–108 (1933); Fund. Math. 21, 129–150 (1933)] en introduisant pour les fonctions F, continûment différentiables dans le domaine S de l'espace euclidien R^k, le produit scalaire $(F_1, F_2) = \int_S (\text{grad } F_1, \text{grad } F_2) dv$ (d'où la norme $\|F\| = (F, F)^{\frac{1}{2}}$) et montrant, comme Nikodým, par les propriétés de l'espace de Hilbert, qu'il existe parmi les fonctions harmoniques H du type F à norme finie, une fonction unique H^0 minimisant $\|F-H\|$, et caractérisée aussi comme fonction unique du type H satisfaisant à $(F-H^0, H) = 0$ quel que soit H. Cela rentre, aussi que l'indication d'extension à des fonctions continûment différentiables par morceaux, dans la théorie des fonctions de classe (BL), introduits par B. Levi et étudiées par Nikodým [loc. cit.] et Deny [Acta Math. 82, 107–183 (1950); ces Rev. 12, 98]. Mais lorsque la frontière de S est formée d'un nombre fini de sphères B_h, l'auteur montre de plus (ce qui n'est peut-être pas conséquence immédiate des traveaux de Nikodým et Deny) que

F considérée au voisinage de chaque B_k comme fonction sur la sphère unité dépendant du paramètre r (rayon d'une sphère concentrique) tend en moyenne quadratique, quand $r \to R$ (rayon de B_k), vers une limite f dite fonction-frontière de F, et que H^0 (de norme au plus égale à celle de F, et égale seulement si $H^0 = F$) est la seule H admettant f comme fonction-frontière. *M. Brelot* (Grenoble).

Parreau, Michel. Sur les moyennes des fonctions harmoniques et la classification des surfaces de Riemann. C. R. Acad. Sci. Paris **231**, 679–681 (1950).

Announcement of results extending a theorem of Nevanlinna [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **57** (1949); these Rev. **11**, 516] to means of order α of harmonic functions on a Riemann surface. The results are applied to the problem of classification of Riemann surfaces. *M. Heins* (Providence, R. I.).

Heins, Maurice. A lemma on positive harmonic functions. Ann. of Math. (2) **52**, 568–573 (1950).

Soit Q l'ensemble des fonctions harmoniques positives $u(m)$ définies sur une surface de Riemann F et normalisées par $u(q)=1$ [q étant un point de F]. Soient $\lambda_i(u)$ ($i=1, \cdots, m$) m fonctionnelles linéaires satisfaisant à $\lambda_i(u_n) \to \lambda_i(u)$ quand $u_n \to u$. Soit K l'image de Q dans la transformation $u \to \Lambda(u) = [\lambda_1(u), \cdots, \lambda_m(u)]$. L'auteur démontre le résultat général suivant: Tout élément extrémal de l'espace convexe K est l'image d'un élément minimal de Q. Il en résulte immédiatement que tout point de K est l'image d'un élément de Q qui est la somme de $m+1$ fonctions minimales au plus (on dit que $u \in Q$ est minimale si toute fonction v harmonique positive dans F satisfaisant à $v(p) \leqq u(p)$ dans F est proportionnelle à u). En appliquant ce lemme à $\lambda_n(u) = u(p_n)$ où p_n est une suite de points dense dans le voisinage d'un point de F, on retrouve très simplement la représentation de R. S. Martin [Trans. Amer. Math. Soc. **49**, 137–172 (1941); ces Rev. **2**, 292] d'une fonction harmonique positive dans F au moyen d'une intégrale de Radon-Stieltjes. En appliquant ce lemme à $\lambda(u) = \int_\gamma dv$, où v est la fonction conjuguée de u et γ un chemin tracé dans F, on obtient une démonstration rapide du résultat fondamental de P. R. Garabedian [ibid. **67**, 1–35 (1949); ces Rev. **11**, 340] dans le problème d'interpolation de Pick-Nevanlinna. *J. Lelong-Ferrand* (Lille).

Nevanlinna, Rolf. Über die Anwendung einer Klasse von Integralgleichungen für Existenzbeweise in der Potentialtheorie. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 146–160 (1950).

This paper is concerned with the connection between the alternating method of Schwarz and Neumann and the theory of integral equations. The following problem is discussed from this point of view. Let A and B denote Jordan regions (not necessarily compact) of a Riemann surface F ($A \cup B = F$) such that the relative boundaries α, β of A, B, respectively, are disjoint. Let A_0, B_0 denote two regions of F which separately contain $A \cap B$ and are such that $A \cap A_0$ and $B \cap B_0$ are connected. Let a be harmonic on $A \cap A_0$ and continuous at each point of $\alpha \cup \beta$, and let b be harmonic in $B \cap B_0$ and continuous on $\alpha \cup \beta$. It is required to construct a function f harmonic in $A_0 \cup B_0$ so that $f-a$ admits a bounded harmonic extension in A and $f-b$ admits a harmonic bounded extension in B. The problem and its connection with the theory of integral equations are studied with the aid of the author's theory of harmonic measure. *M. Heins* (Providence, R. I.).

Keller, Heinrich. Sur la croissance des fonctions harmoniques s'annulant sur la frontière d'un domaine non borné. C. R. Acad. Sci. Paris **231**, 266–267 (1950).

Using a method due to Carleman [same C. R. **196**, 995–997 (1933)] the author has previously studied the growth of a harmonic function u vanishing continuously on the boundary of an infinite domain G of three-dimensional space, considering the behavior of u on sections of G by parallel planes [ibid. **228**, 887–888 (1949); these Rev. **10**, 533]. The present paper gives, without proof, corresponding results for sections by spheres with fixed center O. The case in which G is a half-cone with vertex at O and that in which G is the whole space present exceptional features. *F. W. Perkins* (Hanover, N. H.).

Bertolini, Fernando. Sopra una classe di funzioni armoniche in uno strato cilindrico. Ann. Scuola Norm. Super. Pisa (3) **4**, 101–129 = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **278** (1950).

Let ρ, θ, z be cylindrical coordinates in space, and let $S_{r,R}$ be the infinite cylindrical shell $0 \leqq r < \rho < R \leqq \infty$. Let U be a class of functions $u(\rho, \theta, z)$ harmonic in $S_{r,R}$, and L a linear operator such that $L[u]$ is a function only of θ, z, and such that in $\int_{-\infty}^{\infty} e^{i\omega z} L[u(\rho, \theta, z)]d\theta$, the L may be brought to the outside. The problem considered is this: Given two such operators L_1 and L_2, and two functions $f_1(\theta, z)$ and $f_2(\theta, z)$, to find a function of U such that $L_k[u(\rho, \theta, z)] = f_k(\theta, z)$. The Dirichlet and von Neumann problems are, under suitable hypotheses, special cases of this problem, letting $L_1 = \lim_{\rho \to R} u(\rho, \theta, z)$, etc. The author finds explicit expressions for the solutions of the problem, in case any exist, in terms of Fourier series in θ, Fourier integrals in z, and Bessel functions in ρ, the technique being essentially separation of variables. The functions appearing in the integrals for the solution are obtained by solving a set of auxiliary functional equations involving L_k and f_k. The latter portion of the paper is devoted to a study of some of the special functions appearing in the integral representations in order to prove a decomposition theorem for certain functions harmonic in $S_{r,R}$, namely, that they are the sum of two functions, one harmonic inside the large cylinder and the other harmonic outside the smaller. *J. W. Green* (Los Angeles, Calif.).

Carlson, B. C., and Rushbrooke, G. S. On the expansion of a Coulomb potential in spherical harmonics. Proc. Cambridge Philos. Soc. **46**, 626–633 (1950).

Let r_{12} be the mutual distance of two points P_1 and P_2 in 3-dimensional space. If P_1 and P_2 are referred to the same system of spherical polar coordinates, the expansion of $1/r_{12}$ in a series of spherical harmonics of the coordinates P_1 and P_2 is well known. In the theory of intermolecular forces it is of advantage to have a similar expansion in case P_1 and P_2 are referred to two different systems of spherical polars. The authors give two such expansions. The first assumes that the polar axes coincide; the second assumes parallel polar axes. *A. Erdélyi* (Pasadena, Calif.).

Sadowsky, M. A., and Sternberg, E. Elliptic integral representation of axially symmetric flows. Quart. Appl. Math. **8**, 113–126 (1950).

The results of this paper are complementary to those of Weinstein [same Quart. **5**, 429–444 (1948); these Rev. **10**, 116], on axially symmetric potential flows of an ideal, incompressible fluid. Weinstein based his analysis for the potentials, stream functions, and velocity components of

various axially symmetric distributions of sources on the classical representation in terms of improper integrals involving Bessel functions, while these authors derive these same functions for basic axially symmetric distributions of sources or vorticity over the circumference or the interior of a circle in terms of elliptic integrals of the first or second kind.

The first part of the paper, dealing with the axially symmetric distributions of sources, overlaps to some extent the work of Van Tuyl [ibid. 7, 399–409 (1950); these Rev. 11, 474], who transformed the Bessel integrals given by Weinstein to expressions in terms of elliptic integrals. However, this approach is quite different and in some ways more direct since the Bessel integrals are not introduced. The corresponding results in the two papers are easily seen to agree. This paper does not include a study of the various body shapes and associated stream line patterns resulting from the superposition of the axially symmetric flows with appropriately chosen uniform streams, but it is said that such a study is being carried out in cooperation with others. Further insight is gained, of course, into the cyclic character of Stokes' stream function and these results are in complete agreement with those of Weinstein [loc. cit.] who originally clarified this aspect in connection with the stream function for the source ring. The advantage of the elliptic integral representation is of course obvious. In addition to the results for a source ring and source disc the authors derive a relationship between the potential function for a vortex ring and the stream function for a source ring, and thence are able to determine the potential, stream function, and velocity components for a vortex ring. In contrast to the results for a source ring, the stream function remains single-valued while the potential becomes cyclic, having a circulation equal to the circulation of the vortex ring as one would expect. By integration the corresponding results for a vortex disc are derived from those of a vortex ring.

R. M. Morris (Cardiff).

Differential Equations

Zwirner, Giuseppe. Criteri di esistenza per un problema al contorno relativo all'equazione $y'=f(x, y; \lambda)$. Rend. Sem. Mat. Univ. Padova 19, 141–158 (1950).

The author gives six different sets of conditions under which the boundary value problem $y'(x)=f(x, y(x); \lambda)$, $y(x_0)=y_0$, $y(x_1)=y_1$, has at least one solution. These sets of conditions are complicated, and unsuited for restatement in a review. An existence theorem for the boundary value problem $y'(x)=f(x, y(x); \lambda)$, $y(x_0)=y_0$, $y'(x_1)=v$, is given also.

L. A. MacColl (New York, N. Y.).

Rapoport, I. M. On the boundary problem for a system of linear differential equations with variable coefficients. Doklady Akad. Nauk SSSR (N.S.) 73, 1133–1135 (1950). (Russian)

The author investigates the problem of determining the characteristic values associated with the two-point boundary problem: $dx/dt=\lambda A(t)x$, $0\leqq t\leqq T$, $Px(0)+Qx(T)=0$, where x is an n-dimensional column vector and $A(t)$, P, and Q are $n\times n$ matrices. The method of expanding the solution as a power series in λ is applied. No mention is made of the results of G. D. Birkhoff and Langer [Proc. Amer. Acad. Arts Sci. 58, 51–128 (1923)] obtained in connection with the above equation.

R. Bellman.

Gradštein, I. S. Linear equations with variable coefficients and small parameters in the highest derivatives. Mat. Sbornik N.S. 27(69), 47–68 (1950). (Russian)

A perturbed differential equation

$$(1)\qquad \sum_{k=0}^{m}\alpha_k(t, \eta)\frac{d^kX}{dt^k}+\eta\sum_{k=1}^{\mu}\eta^{k-1}\alpha_{m+k}(t, \eta)\frac{d^{m+k}X}{dt^{m+k}}=\alpha_{-1}(t, \eta)$$

and a degenerate equation

$$(2)\qquad \sum_{k=0}^{m}\alpha_k(t, 0)\frac{d^kx}{dt^k}=\alpha_{-1}(t, 0)$$

are considered, where each α is analytic in η in the domain $t_0\leqq t\leqq T$, $0\leqq\eta\leqq\epsilon_0$. By hypothesis the α_{ks} in the expansion

$$\alpha_k(t, \eta)=\sum_{s=0}^{\infty}\eta^s\alpha_{ks}(t), \qquad k=-1, 0, 1, \cdots, m+\mu,$$

possess derivatives of a suitably high order; $\alpha_{m0}(t)\neq 0$; $\alpha_{m+\mu, 0}(t)\neq 0$; and the roots $\omega_i(t)$, other than the zero root of multiplicity m, of the characteristic equation $\sum_{k=0}^{\mu}\alpha_{m+k, 0}(t)\omega^{m+k}=0$ are distinct with nonpositive real parts and in some interval $t_0<t<t_1$ the $\omega_i(t)$ are all definitely negative. The objective of this paper is to specify conditions interrelating the initial values at $t=t_0$ and the coefficients of equation (1) so that as $\eta\rightarrow+0$ a solution $X(t, y)$ of (1) and its first $(m-1)$-derivatives will tend on $t_0<t\leqq T$ to a given solution $x(t)$ of (2) and its respective derivatives. For example, if $X^{(k)}(t_0, \eta)=O(\eta^0)$, $k=0, 1, \cdots, m$, $X^{(k)}(t_0, \eta)=O(\eta^{m-k})$, $k=m, \cdots, m+\mu-1$, and if $x^{(k)}(t_0)=\lim_{\eta\rightarrow+0}X^{(k)}(t_0, \eta)$, $k=0, \cdots, m-1$, then uniformly $\lim_{\eta\rightarrow+0}X^{(k)}(t, \eta)=x^{(k)}(t)$, $t_0\leqq t\leqq T$, $k=0, \cdots, m-1$.

More generally, if $X(t, \eta)$ is a solution of (1) with $X^{(k)}(t_0, \eta)=O(\eta^{-k})$, $k=0, \cdots, m+\mu-1$, and $x(t)$ is a solution of (2), in order that $\lim_{\eta\rightarrow+0}d^kX/dt^k=d^kx/dt^k$ on $t_0<t\leqq T$, $k=0, \cdots, m-1$, it is necessary and sufficient that the initial values of $X(t, \eta)$ and $x(t)$ be connected by the relations $\Phi_r(t_0, 0)=0$, $r=1, \cdots, m$. The precise form of the Φ_r as given in the paper is too long to reproduce here; let it suffice to say that the Φ_r are linear in the $x^{(k)}(t_0)$, $k=0, \cdots, m-1$, linear in the $X^{(k)}(t_0, \eta)$, $k=0, \cdots, m+\mu-1$, and involve the α's and certain of their derivatives appearing in the sums $\sum_{s=0}^{m-1}\alpha_{ks}(t)\eta^s$, $k=m, \cdots, m+\mu$ and $\sum_{s=0}^{k-1}\alpha_{ks}(t)\eta^s$, $k=1, \cdots, m$. Also the m relations $\Phi_r(t_0, 0)=0$ can be solved for the $x^{(k)}(t_0)$. The paper utilizes the work of Noaillon [Mém. Soc. Roy. Sci. Liège (3) 9, no. 15 (1912)]. See also in this regard Turrittin [Amer. J. Math. 58, 364–376 (1936)] and Trjitzinsky [Acta Math. 67, 1–50 (1936)].

G. A. Hufford and H. L. Turrittin (Princeton, N. J.).

Holmberg, Bengt. On some phase-potential relations. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 20, 49–57 (1950).

The author relates his extension of Hulthén's potential-asymptotic phase results to the case of noncentral interactions [same journal 18, 88–117 (1948)] to results of Møller and generalizes Fröberg's results [Ark. Mat. Astr. Fys. 36A, no. 11 (1949); these Rev. 11, 249] to noncentral interactions.

N. Levinson (Cambridge, Mass.).

Sansone, G. Sopra l'equazione di A. Liénard delle oscillazioni di rilassamento. Ann. Mat. Pura Appl. (4) 28, 153–181 (1949).

The author gives sufficient conditions for

$$(*)\qquad i''+f(i)i'+i=0,$$

where $f(i)$ is continuous, to have a periodic solution (other

than $i=0$). He proves the existence of a unique periodic solution under two hypotheses. Hypothesis I: (1) $f(i)<0$ for $\delta_{-1}<i<\delta_1$, where $\delta_{-1}<0<\delta_1$ and $f(i)>0$ for $i<\delta_{-1}$ and $i>\delta_1$; (2) let $F(i)=\int_0^i f(i)di$ and let there exist $\Delta>0$ so that $F(\Delta)=F(-\Delta)=0$; (3) let $F(i)\to\infty$ as $i\to\infty$ (or let $F(i)\to-\infty$ as $i\to-\infty$). This is more general than the result of Liénard [Rev. Gen. Électricité **23**, 901–912, 946–954 (1928)] in that $f(i)$ is not required to be odd but is less general in that Liénard requires only $F(i)<0$, $0<i<\Delta$, $f(i)>0$, $i>\Delta$, in place of (1). Hypothesis II: $f(i)<0$, $-\delta<i<\delta$, $f(i)>0$, $|i|>\delta$, and (3) as above. In case $f'(i)$ exists, hypothesis II is a special case of a result of Levinson and Smith [Duke Math. J. **9**, 382–403 (1942); these Rev. **4**, 42]. [One case of the equation of Levinson and Smith [loc. cit., theorem III] is $i''+f(i)i'+g(i)=0$, and for uniqueness it is assumed $f'(i)$ and $g'(i)$ exist. However, in the special case (*) considered by the author where $g(i)\equiv i$, the solutions of equation (*) are in one-to-one correspondence with solutions of $x''+F(x')+x=0$, where $F'(i)=f(i)$ and $x'=i$. Uniqueness for this equation is immediate with f continuous, and this implies also uniqueness for the equation (*). Thus the case hypothesis II of the author is a consequence of the result of Levinson and Smith.]
N. Levinson (Cambridge, Mass.).

Železcov, N. A. On the theory of the symmetric multivibrator. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **20**, 788–797 (1950). (Russian)

The symmetrical multivibrator of Abraham-Bloch has been discussed in sketchy manner by Andronow and Chaikin [Theory of Oscillations, Princeton University Press, 1949, p. 274; these Rev. **10**, 535]. The basic equations may be reduced, with suitable choice of the variables x, y and time unit t to the form of two symmetrical equations in x and y of which one is $\dot{x}=(x-k\varphi'(y)y)/(k^2\varphi'(x)\varphi'(y)-1)$. The function $\varphi(x)$ is the characteristic of the vacuum tubes. The author assumes for it a piecewise linear approximation such that $\varphi'(x)=0$ for $|x|>1$ and $\varphi'(x)=1$ for $|x|<1$. He studies [as in loc. cit.] the discontinuous oscillations, which arise only when $k<1$. As a consequence, in the Andronow and Chaikin diagram [loc. cit., fig. 248] the two ovals become polygons, the interior one being a square with sides parallel to the axes. Under these simplifying assumptions it is shown that there is a unique discontinuous oscillation and that it is stable [cf. Andronow and Witt, C. R. (Doklady) Acad. Sci. URSS **1930**, 189–192]. *S. Lefschetz.*

Wasow, Wolfgang. A study of the solutions of the differential equation $y^{(4)}+\lambda^2(xy''+y)=0$ for large values of λ. Ann. of Math. (2) **52**, 350–361 (1950).

This paper is concerned with the asymptotic solution of the special differential equation in the title, in which λ is a large real parameter and x is complex. The equation is of a type which is important in hydrodynamics. Solutions of the equation in the form of Laplace contour integrals are easily obtained. With special choices of the paths seven solutions are singled out. Asymptotic evaluations of them are obtained by the method of steepest descent. Forms for the solutions are thus deduced which apply when $|x|\geqq|x_0|>0$, when $|\xi|\geqq N$, and when $|\xi|\leqq N$, where N is an arbitrary constant and $\xi=\frac{2}{3}\lambda(-x)^{3/2}$. *R. E. Langer.*

Sarginson, K. An operational method for determining the series solution of a linear differential equation of rank two. Math. Gaz. **34**, 8–10 (1950).

Let $f(\delta)$ and $g(\delta)$ be polynomials in the operator $\delta=xd/dx$. The author derives solutions of the differential equation $[f(\delta)-xg(\delta)]y=0$ in the three cases: (1) when $f(z)$ has distinct zeros no two of which differ by an integer; (2) when $f(z)$ has some repeated linear factors; and (3) when some of the zeros of $f(z)$ differ by integers. The intervals of convergence of the series obtained for y are not determined. The operational method consists of using differential operators and their inverses. *R. V. Churchill.*

Moore, John R. The generalized response of linear systems for arbitrary initial conditions. J. Appl. Phys. **21**, 933–935 (1950).

A formula is derived for the function $\theta(t)$ that satisfies the differential equation $G(p)\theta(t)=H(p)i(t)$ and any prescribed initial conditions. The functions G and H are polynomials in the operator $p=d/dt$, with constant coefficients, and $i(t)$ is any prescribed input function. The linear factors of G are assumed to be known and distinct.
R. V. Churchill (Ann Arbor, Mich.).

Rogers, T. A., and Hurty, W. C. Relay servomechanisms. The shunt-motor servo with inertia load. Trans. A.S.M.E. **72**, 1163–1172 (1950).

In the first part of this paper the motion of a shunt-motor relay servomechanism, subject to an input which is either constant or proportional to time, is discussed along conventional lines. Considerable attention is paid to the selection of values of the parameters which shall be representative of actual systems. The latter part of the paper is devoted to a study of the case in which the input is a sinusoidal function of time. Phase-plane diagrams were obtained by means of a mechanical differential analyzer, and these curves are used as the basis for an extensive discussion of the errors of the servomechanism. *L. A. MacColl* (New York, N. Y.).

Sears, D. B., and Titchmarsh, E. C. Some eigenfunction formulae. Quart. J. Math., Oxford Ser. (2) **1**, 165–175 (1950).

The following quotation from the paper summarizes it adequately: "The object of this paper is to correct § 4.14 and § 5.8 of Titchmarsh, Eigenfunction Expansions Associated with Second-Order Differential Equations [Oxford, 1946; these Rev. **8**, 458]. The example given there is inconsistent with the general theorem, and actually both are incorrect. It is a question of the spectrum associated with the equation $d^2\varphi/dx^2+\{\lambda-q(x)\}\varphi=0$ $(0\leqq x<\infty)$, in the case where $q'(x)=O\{|q(x)|^c\}$ $(0<c<\frac{3}{2})$, $q(x)\to-\infty$, $q''(x)$ is ultimately of one sign, and $\int^\infty|q(x)|^{-1/2}dx$ is convergent. (The condition $q(x)\leqq 0$ imposed is not really relevant.) The analysis given in § 5.8 is correct so far as positive values of λ are concerned, and the result stated, that the spectrum is discrete for $\lambda>0$, is correct. The actual result, however, is that under the above conditions the whole spectrum is discrete (and not that it is continuous in $(-\infty, 0)$, as is stated in the book). The mistake arises in the first formula on p. 109. This contains the function $\xi(t)=\xi(t,\lambda)=\int_0^t\{\lambda-q(x)\}^{1/2}dx$, which may have a branch-point at $\lambda=q(0)$. The conclusion that $\mu(\lambda)$ is an integral function is therefore false, and the argument based on it fails. The main mistake in § 4.14 is that the discrete spectrum arising from the zeros of $\sin(i\pi\lambda^{1/2})$ is ignored." *R. E. Langer* (Madison, Wis.).

Doyle, Thomas C. Invariant theory of the general ordinary, linear, homogeneous, second order, differential boundary problem. Duke Math. J. **17**, 249–261 (1950).

The paper concerns the boundary problem associated with the differential equation $y''+p(x)y'+[\lambda q(x)+r(x)]y=0$,

with $q(x)\neq 0$, that was dealt with by the reviewer [Amer. Math. Monthly 54, no. 7, part II (1947); these Rev. 9, 87]. Under the transformation group $y=e^{\sigma(x)}y^*$, $x^*=\xi(x)$, with $\xi'(x)\neq 0$, the form of the differential equation is preserved. The matter at issue is a discussion of the boundary problem in an invariant form. *R. E. Langer* (Madison, Wis.).

Petrescu, Şt. On point invariants of the differential equation $y'''=F(x,y,y',y'')$. Acad. Repub. Pop. Române Bul. Şti. Ser. A. 1, 361–368 (1949). (Romanian. Russian and French summaries)

The author seeks the invariants of $y'''=F(x,y,y',y'')$ under the group $\bar{x}=\bar{x}(x)$, $\bar{y}=\lambda(x)y+\mu(x)$. He proceeds by the method of Vranceanu replacing the problem by that of the equivalence of a set of Pfaffians under the group induced upon them by the above. Integrability conditions of the equivalence problem lead to certain quantities I, J, K (the first a relative invariant) which play leading roles. Various cases arise depending on the vanishing of I, J, K, and their derivatives with respect to the Pfaffians, leading to special forms of the original differential equation and to specific invariants. In certain of these cases the equivalence problem assumes a relatively simple form. *J. L. Vanderslice.*

Petrescu, Şt. On investigations of the point invariants of the differential equation $y'''=F(x,y,y')$. Acad. Repub. Pop. Române Bul. Şti. Ser. A. 1, 433–438 (1949). (Romanian. Russian and French summaries)

This note is a sequel to the paper reviewed above and considers the special case $y'''=F(x,y,y')$. The methods are the same and the results analogous. *J. L. Vanderslice.*

***Bernstein, Dorothy L. Existence Theorems in Partial Differential Equations.** Annals of Mathematics Studies, no. 23. Princeton University Press, Princeton, N. J., 1950. ix+228 pp. $2.50.

Cet ouvrage tire son origine d'une vaste enquête sur les problèmes de calcul par machines, menée par l'"Engineering Research Associates, Inc." pour l'"Office of Naval Research." Tompkins, qui en avait été chargé, rappelle dans sa préface certains faits parfois méconnus lorsqu'on substitue à une équation aux dérivées partielles une équation aux différences ordinaire, puis introduit le travail de l'auteur. Il s'agit d'un exposé d'ensemble des théorèmes d'existence relatifs aux systèmes différentiels, où l'on se préoccupe surtout des possibilités d'application effective au calcul, mais sans se borner étroitement à ce point de vue. Cet exposé comprend quatre chapitres: introduction (18 pp.); problème de Cauchy pour équation et systèmes d'équations du premier ordre (66 pp.); équations de deuxième ordre (113 pp.); équations d'ordre supérieur (19 pp.). Peut-être pourrait-on regretter que les systèmes du premier ordre à plusieurs fonctions inconnues n'aient pas été rattachés plutôt aux équations d'ordre deux au moins: on sait en effet que la nature réelle ou imaginaire des multiplicités caractéristiques est essentiellement à mettre en évidence pour ces systèmes, et que, à moins de se limiter aux données analytiques, on ne peut espérer obtenir des résultats indépendants d'une telle classification. La bibliographie (11 pp.) qui termine le volume n'a pas, bien entendu, la prétention d'épuiser le sujet; elle pourra, comme l'ensemble de l'ouvrage, rendre de grands services; par l'intermédiaire de certains des articles qui y sont cités, on pourra retrouver les titres d'ouvrages intéressants que l'auteur n'a pas jugé utile de mentionner explicitement. *M. Janet* (Paris).

Pfeiffer, G. La réception et l'intégration par la méthode spéciale des équations, systèmes d'équations semi-Jacobiens, des équations, systèmes d'équations semi-Jacobiens généralisés aux dérivées partielles du premier ordre de plusieurs fonctions inconnues. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. 1946, no. 8, 153–162 (1947). (Ukrainian. Russian and French summaries)

As the title indicates, the paper is concerned with a special method for solving semi-Jacobian systems of partial differential equations. Such a system arises from a linear partial differential equation of the first order containing one or more parameters whose elimination leads to the Jacobian system of equations which are nonlinear. Three illustrative examples are worked out, but the validity of the stated "rules" is not clear. *M. S. Knebelman* (Pullman, Wash.).

Moisil, Gr. C. Sur les systèmes d'équations aux dérivées partielles linéaires et à coefficients constants. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 341–351 (1949). (Romanian, Russian, and French)

This note states, without proofs, results on systems of linear partial differential equations, in several unknowns, with constant coefficients. The results deal with finite bases for infinite systems, with the derivation, from a system in given unknowns, of equations free of certain unknowns, and with the transformation of the unknowns in a system. *J. F. Ritt* (New York, N. Y.).

Moisil, Gr. C. Systèmes d'équations aux dérivées partielles et idéaux de polynomes. An. Acad. Repub. Pop. Române. Secţ. Şti. Mat. Fiz. Chim. Ser. A. 2, no. 17, 467–476 (1949). (Romanian. Russian and French summaries)

This is an algebraic study of systems of linear partial differential equations in one unknown, $P_i(\partial/\partial x_1,\cdots,\partial/\partial x_n)\varphi=0$, $i=1,\cdots,r$, with each $P_i(X_1,\cdots,X_n)$ a polynomial with constant coefficients. The author considers the polynomial ideal $(P_1,\cdots,P_r)$ and carries over, into the language of differential equations, certain notions and theorems of ideal theory. Thus he considers sums, products, quotients, and intersections of ideals, complementary ideals, prime ideals, and irreducible ideals. *J. F. Ritt* (New York, N. Y.).

Colino, Antonio. A study of the excitation of waves. Revista Acad. Ci. Madrid 43, 273–285 (1949). (Spanish)

In the nonhomogeneous scalar wave equation $(\nabla^2+k^2)\psi=q$, the term q may be considered as exciting the wave. The author derives formulas in terms of q for the coefficients in the expansion of ψ in a series of orthogonal solutions of the corresponding homogeneous wave equation. These results are applied in particular to the solutions found by separation of variables in cylindrical and spherical coordinates. The case of electromagnetic waves is also studied. In the Maxwell equation $\nabla\times H=\partial D/\partial t+I$, the term I, which is the conduction current density vector, is considered as exciting the wave. Formulas are given for E and H waves (also called transverse magnetic and transverse electric in wave guide theory) in terms of I, in particular for the case of cylindrical coordinates. *O. Frink* (State College, Pa.).

Kahan, Théo. Une méthode variationnelle propre à l'étude de l'équation des ondes. Revue Sci. 87, 205–211 (1949).

Die Untersuchungen von Robin über das Dirichletsche Problem einerseits und die Methode von Ritz zur Auflösung von Randwertaufgaben andererseits, über die der Verfasser zu Beginn seiner Arbeit referiert, veranlassten ihn, die fol-

gende Randwertaufgabe für die Wellengleichung $\Delta\psi+k^2\psi=0$ in nachstehender Weise in Angriff zu nehmen: Es sei gegeben ein endlicher oder unendlicher Bereich D, in dem eine gewisse Anzahl von Quellen $Q_1, \cdots, Q_n$ gegeben sind und die Lösung der obigen Gleichung habe an verschiedenen Grenzen $S_1, S_2, \cdots$ vorgeschriebene Randwerte. Durch das Superpositionsprinzip verwandelt sich die obige Aufgabe in die Aufgabe, eine Quellendichtebelegung für die einzelnen Oberflächen zu finden, die der Integralgleichung $\int G(m, M)\rho(m)d\sigma_m+V_0(M)=0$ genügt. Dabei bedeutet $G(m, M)=e^{ikr(m,M)}/r(m, M)$ eine gegebene Funktion auf der Oberfläche und $\rho(m)$ die gesuchte Quellendichte. Diese Integralgleichung veranlasst entsprechend den Ideen von Robin zur Betrachtung der folgenden Rekursionsformel $V_1(M)=\int G(m, M)fd\sigma_m$, $V_i(M)=\int G(m, M)V_{i-1}(m)d\sigma_m$ $(i=2, 3, \cdots)$, wobei $f=V_0$. Nun wird für eine Annäherung an die gesuchte Quellendichte entsprechend der Ritzschen Methode der Ansatz $\rho_n=a_1f+a_2V_1+\cdots+a_{n+1}V_n$ gemacht und die Konstanten $a_1, \cdots, a_n$ entsprechend der Forderung $\int|E_n|^2d\sigma_m\rightarrow$minimum bestimmt, wobei

$$E_n=V_0+a_1V_1+\cdots+a_nV_n.$$

P. Funk (Wien).

Kupradze, V. D. On the boundary value problems of the steady vibrations of elastic bodies. Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 190–193 (1950). (Russian)

The author considers steady vibration problems consisting of the system of equations $\Delta^*\bar{u}+k_2^2\bar{u}=0$ (in the interior or the exterior of a smooth surface S, where $\Delta^*\equiv\Delta+(\lambda+\mu)\mu^{-1}$ grad div, $k_2^2=\sigma(\omega^2/\mu)$, $\bar{u}=(u_1, u_2, u_3)$ is the displacement vector, λ and μ are Lamé's constants of elasticity, σ is the density, and ω is the frequency of vibration), subject to the boundary conditions $\alpha\bar{u}_{i,a}+\beta T_{i,a}=\bar{f}$ on S, where the subscripts i, a refer to the inner and outer problems, respectively, T is the normal stress on S, $\bar{f}$ is given on S, and the given functions α and β are such that on S either: (1) $\alpha\neq0$, $\beta=0$, (2) $\alpha=0$, $\beta\neq0$, or (3) $\alpha\neq0$, $\beta=0$ on a subset of S and $\alpha=0$, $\beta\neq0$ on the remainder of S. A condition at infinity is imposed on $\bar{u}$ in the outer problem. There are six vibration problems in all. It is announced that all these vibration problems may be translated into equivalent integral equations to which the Fredholm theory is applicable, by first introducing certain potentials of surface distributions (of single, double, and antenna layers), and using the "jump conditions" at the surface occupied by the masses to obtain the equivalent integral equations.

J. B. Diaz (College Park, Md.).

Datzeff, A. Sur le problème de la propagation des ondes. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **40**, 173–259 (1944); **41**, 231–245 (1945). (Bulgarian. French summary)

"La méthode de résolution développée ici, rassemble à une autre, exposée dans notre thèse [Ann. Physique **10**, 583–673 (1938)]. Les deux méthodes sont également applicables au cas de problème stationnaire (onde plane incidente illimitée), mais celui-ci a l'avantage de s'appliquer immédiatement pour le problème non stationnaire et se prête à généralisation au cas de deux ou trois variables indépendentes."

From the author's summary.

Datzeff, A. Sur le refroidissement d'un corps non homogène. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **43**, 1–15 (1947). (Bulgarian. French summary)

[Volume number misprinted 42 on title page.]

Datzeff, Assène. Sur le refroidissement d'une barre se composant de deux barres homogènes limitées. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **43**, 113–135 (1947). (Bulgarian. French summary)

[Volume number misprinted 42 on title page.]

Dacev, A. B. On the linear problem of Stefan. The case of two phases of infinite thickness. Doklady Akad. Nauk SSSR (N.S.) **74**, 445–448 (1950). (Russian)

The author considers the system of partial differential equations $a^2\partial^2u/\partial x^2=\partial u/\partial t$, $x'<x<s(t)$, $b^2\partial^2v/\partial x^2=\partial v/\partial t$, $s(t)<x<x''$, subject to certain initial conditions. The system was first treated by Stefan in connection with problems in heat conduction, and various cases have subsequently been discussed by Rubinstein [Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **11**, 37–54 (1947); these Rev. **8**, 516] and the author [same Doklady (N.S.) **58**, 563–566 (1947); these Rev. **9**, 513]. The author divides a finite t-interval into n parts, over which the initial functions are assumed independent of t, finds the solution in each subinterval by an iteration method, and shows that as $n\rightarrow\infty$ the pieced-together solution converges to a solution of the original equation.

R. Bellman.

Evans, G. W., II, Isaacson, E., and MacDonald, J. K. L. Stefan-like problems. Quart. Appl. Math. **8**, 312–319 (1950).

If an infinite metal slab of any thickness L, $0<L<\infty$ of uniform crystalline structure is heated from the front face through a critical temperature, new crystals appear on the heated face. If the front face is planar and is heated uniformly and the back face is insulated, the interface between the two types of crystals is planar. This paper is concerned with the determination of the history of this interface. The first problem considered is the case in which the slab has been heated to the critical temperature, 0, and then a constant heat source applied to the front face of the slab. The authors find the equation of the interface $x=x(t)$ where the temperature distribution $u(x, t)$ satisfies the equations $u_t=\alpha^2u_{xx}$ for $0<x<x(t)$, $u=0$ for all $x\geqq x(t)$, where the coefficient of thermal diffusivity α^2 is assumed to be constant. The boundary conditions are $A\dot{x}(t)=u_x[x(t), t]$, $x(0)=0$, $u_x(0, t)=g$, where A and g are constants. The method of solution is based on the assumption that u is expressible as a power series about $x=0$, $t=0$ for all x in $0\leqq x\leqq x(t)$. Then the preceding relations are used to determine all derivatives of $x(t)$ at the origin so that $x(t)$ may be written in a Taylor series. In the case in which g is an analytic function of time a similar expression for $x(t)$ is given which reduces to the preceding case when $g(t)$ is made constant.

The method of solution of the above problem is generalized to include the case in which there are nonanalytic boundary and initial conditions. The Laplace transform is applied to the preceding problem with g as a function of t and the transformed problem is solved. This in turn leads to an integral equation for the determination of the equation of the interface in the form $t=f(x)$. Once this result is obtained, it may be used in the inverse transform of the solution to get the temperature distribution in that region which has been recrystallized. A third problem in which the thickness of the slab is allowed to be infinite is also discussed. A discussion of the degree of agreement of the equations obtained by taking the first five sums of the terms of the series representation of the equation of the interface for the first problem is given for data which are approximately that

of a cast iron. A comparison of the approximate equations for the interfaces based on the first two terms of the expansions of $x(t)$ in each of the three problems is made. The results of both of these discussions are shown graphically.

C. G. Maple (Washington, D. C.).

Danckwerts, P. V. Unsteady-state diffusion or heat-conduction with moving boundary. Trans. Faraday Soc. **46**, 701–712 (1950).

A general method of solution is described for a class of problems in unsteady linear heat conduction or diffusion which involve two phases or regions separated by a moving interface. The method is an adaptation of well-known solutions for the conduction of heat to such broad boundary conditions that the following problems are all included as special cases: the absorption by a liquid of a single component from a mixture of gases; tarnishing reactions; condensation of a vapor at the surface of a cold liquid or on a cooled surface; gas reacting at solid surface to form gaseous product; solution of a gas in a liquid, followed by reaction with solute; progressive freezing of a liquid.

H. P. Thielman (Ames, Iowa).

Obrechkoff, N. Sur un problème limite relativement l'équation de la chaleur. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **38**, 303–318 (1942). (Bulgarian. French summary)

Soit $\varphi(x)$ une fonction intégrable au sens de Lebesgue, $\varphi(1)=0$, et désignons par μ_n, $n=0, 1, 2, \cdots$, les racines positives de l'équation $\mu \sin \mu - k \cos \mu = 0$, où k est une constante positive. Alors la série

$$u(x, t) = 2\sum_{n=0}^{\infty} e^{-\mu_n^2 t} \frac{\sin \mu_n(1-x)}{k+\sin^2 \mu_n} \times \left[\varphi(0) \sin \mu_n + k\int_0^1 \varphi(y) \sin \mu_n(1-y)dy \right]$$

est uniformément convergente pour $0 \leqq x \leqq 1$, $t \geqq \epsilon > 0$, pour chaque $\epsilon > 0$, et sa somme $u(x, t)$ satisfait à l'équation $\partial^2 u/\partial x^2 = \partial u/\partial t$ et aux conditions $u(1, t) = 0$,

$$[\partial u/\partial t - u\partial u/\partial x]_{x=0} = 0$$

pour chaque $t>0$. La fonction $u(x, t)$ tend vers $\varphi(x)$ lorsque $t \to 0$ pour chaque point de continuité x. On a aussi un résultat analogue pour le cas où les conditions limites sont les suivantes: $u(0, t) = 0$, $[\partial u/\partial x - hu]_{x=1} = 0$ pour $t>0$.

From the author's summary.

Barenblatt, G. I. On the solution of the equation of heat conduction with a nonhomogeneous boundary condition. Doklady Akad. Nauk SSSR (N.S.) **74**, 201–204 (1950). (Russian)

The author considers two methods of solution of the heat equation $\partial^2 T/\partial x^2 = q(x)\partial T/\partial x$ $(0 \leqq x \leqq \infty)$, where T is subject to the boundary conditions

$$T(0, t) \sin \alpha - \frac{\partial T(0, t)}{\partial x} \cos \alpha = \phi(t); \quad T(x, 0) = f(x);$$

$$\frac{\partial T(\infty, t)}{\partial x} = 0.$$

It is assumed that $q(x)$ satisfies the conditions of an earlier paper [same Doklady (N.S.) **72**, 667–670 (1950); these Rev. **12**, 183]. In both methods the substitution $T = T_1 + T_2$ is made, where T_1 and T_2 each satisfy the differential equation. In the first method T_1 and T_2 satisfy the above boundary conditions with $\phi(t) \equiv 0$ and $f(x) \equiv 0$, respectively. For the second method, the boundary conditions are as follows:

$$T_1(0, t) \sin \alpha - \frac{\partial T_1(0, t)}{\partial x} \cos \alpha = 0; \quad T_1(x, 0) = f(x) - K;$$

$$\frac{\partial T_1(\infty, t)}{\partial x} = 0,$$

$$T_2(0, t) \sin \alpha - \frac{\partial T_2(0, t)}{\partial x} \cos \alpha = \phi(t); \quad T_2(x, 0) = K;$$

$$\frac{\partial T_2(\infty, t)}{\partial x} = 0,$$

where K is some constant. In both instances T_1 and T_2 are constructed by the methods of the previous paper. The special case $q(x) = x^m$ is given as an example.

C. G. Maple (Washington, D. C.).

Jaeger, J. C. Conduction of heat in a solid with a power law of heat transfer at its surface. Proc. Cambridge Philos. Soc. **46**, 634–641 (1950).

The nonlinear boundary value problem $v_t(x, t) = kv_{xx}(x, t)$ $(x<0, t>0)$, $v(x, 0) = v_0$, $-Kv_x(0, t) = H[v(0, t)]^m$, where k, K, v_0, H, and m are constants, is solved formally by first introducing power series in t for the unknown temperature and flux at the surface and then determining the coefficients in those series. In this manner the temperature function is determined as a series of repeated integrals of error functions. The convergence is rapid only for small values of t. The special cases $m=5/4$ and $m=4$, and generalizations of the condition at the surface for which the same method applies, are noted. Surface temperatures are also found by methods of difference equations, where t is not limited to small values. Graphs of these temperatures corresponding to various laws of heat transfer at the surface are shown.

R. V. Churchill (Ann Arbor, Mich.).

Satoh, Tunezo. On the mathematical analysis of the problem of the conduction of heat when emissivity is variable. J. Phys. Soc. Japan **5**, 253–254 (1950).

A solution $u(x, t)$ of the heat equation $u_{xx} = u_t$ $(x>0, t>0)$ subject to the conditions $u(x, +0) = c$, c a constant, $u_x(+0, t) = h(t)u(0+, t)$ is obtained (the given function $h(t)$ subject to certain minor restrictions). Goursat [Cours d'analyse mathématique, vol. 3, 4th ed., Gauthier-Villars, Paris, 1927] presents the solution of a more general problem than the one under review.

F. G. Dressel.

Christov, Chr. On a problem of gas diffusion. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **41**, 143–163 (1945). (Bulgarian. English summary)

Forsythe, George E. Solution of the telegrapher's equation with boundary conditions on only one characteristic. J. Research Nat. Bur. Standards **44**, 89–102 (1950).

The problem is to find a real-valued function $v(x, t)$ defined everywhere on the closed two-dimensional region consisting of all points (x, t) with $-\pi \leqq x \leqq \pi$, $0 \leqq t < \infty$, which under general conditions satisfies the "telegrapher's equation" $v_{xt} + \frac{1}{4}v = 0$ and which for the characteristic $t=0$ reduces to a given function $f(x)$ which is sectionally smooth for $-\pi \leqq x \leqq \pi$ and whose average value is zero. It is shown how to find unique solutions represented in terms of a

Green's function

$$v(x,t)=\pi^{-1}\sum_{k=1}^{K}J_kG(x-x_k,t)+\pi^{-1}\int_{-\pi}^{\pi}G(x-u,t)\xi'(u)du,$$

where the Green's function $G(x, t)$ is the solution of the problem for $f(x)=\sigma_0(x)=\sum_{n=1}^{\infty}n^{-1}\sin nx$, $J_k=f(x_k+0)-f(x_k-0)$ is the discontinuity at the point x_k $(k=1,\cdots,K)$, and $\xi(x)=f(x)-\pi^{-1}\sum_{k=1}^{K}J_k\sigma_0(x-x_k)$ is a continuous function, since the discontinuities have been removed. The Green's function $G(x,z)=\sum_{n=1}^{\infty}n^{-1}\sin(nx+z/n)$ is computed to an accuracy of ±0.001 for $x=-\pi(\pi/36)\pi$ and for 14 z-values up to 36. An alternate solution by a difference equation is mentioned. *S. C. van Veen* (Delft).

Pleijel, Åke. On the eigenvalues and eigenfunctions of elastic plates. Comm. Pure Appl. Math. **3**, 1–10 (1950).

This paper contains some investigations concerning the asymptotic behavior of eigenvalues and eigenfunctions of elastic plates. The equation for the deflection u of a vibrating plate (supposed to occupy a plane domain V possessing a smooth boundary S), (i) $\Delta\Delta u-\lambda u=0$ in V, where $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$, is considered together with the three different boundary conditions on S: (a) clamped edges, $u=0$, $\partial u/\partial n=0$; (b) simply supported edges, $u=0$,

$$I_2(u)\equiv\mu\Delta u+(1-\mu)(n_1^2u_{xx}+2n_1n_2u_{xy}+n_2^2u_{yy})=0;$$

(c) free edges, $I_2(u)=0$,

$$I_3(u)=\partial\Delta u/\partial n-(1-\mu)(\partial/\partial s)[n_1n_2(u_{yy}-u_{xx})+(n_1^2+n_2^2)u_{xy}]=0,$$

where μ is an elastic constant, and n_1, n_2 the components of the outer normal to S. [For the theory of elastic plates and the smoothness assumptions on S reference is made to Friedrichs, Math. Ann. **98**, 205–247 (1927).] The method employed is patterned after that of Carleman [Åttonde Skand. Mat. Kongress, 1934, pp. 34–44 (1935); Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **88**, 119–132 (1936)], which consists in first determining the asymptotic behavior of the compensating part (regular part) of a certain Green's function, and then deriving from this the asymptotic behavior of eigenvalues and eigenfunctions by applying a Tauberian theorem of Hardy and Littlewood. Denote by $G(P,Q;-k^4)$, where P and Q are points of $V+S$, the Green's function of the equation (ii) $\Delta\Delta u+k^4u=0$ in V, k real, subject to one of the boundary conditions (a), (b), or (c). By Mercer's theorem $G(P,Q;-k^4)=\sum_{n=1}^{\infty}\phi_n(P)\phi_n(Q)/(\lambda_n+k^4)$, the series being uniformly convergent, where $\phi_n(x,y)$ is the corresponding eigenfunction of (i) subject to (a), (b), or (c), belonging to the eigenvalue λ_n, and $0\leqq\lambda_1\leqq\lambda_2\leqq\cdots\rightarrow\infty$. Also $G(P,Q;-k^4)=\Gamma(P,Q;-k^4)-\gamma(P,Q;-k^4)$, where $\Gamma(P,Q;-k^4)$ is the elementary solution of (ii), and $\gamma(P,Q;-k^4)$ is the compensating (regular) part of G. For $Q=P$, (iii) $1/8k^2-\gamma(P,P;-k^4)=\sum_{n=1}^{\infty}\phi_n^2(P)/(\lambda_n+k^4)$. Variational methods are employed to prove the following estimate for the compensating part:

$$\gamma(P,P;-k^4)=O(k^{-4}l_P^{-2}+k^{-3}l_P^{-1}),$$

where l_P is the distance from P to the boundary S. The following Tauberian theorem of Hardy and Littlewood [Proc. London Math. Soc. (2) **30**, 23–37 (1929)] is then used: If for $x\rightarrow+\infty$, $\sum_{n=1}^{\infty}A_n/(\lambda_n+x)=Ax^{-\sigma}+o(x^{-\sigma})$, where $A_n>0$, $\lambda_n\uparrow\infty$, $A\neq0$, $0<\sigma<1$, then

$$\sum_{\lambda_n<t}A_n=[A\sin(\sigma\pi)/\sigma\pi]t^{1-\sigma}+o(t^{1-\sigma})$$

as $t\rightarrow+\infty$. Application of this theorem to (iii) gives the result $\sum_{\lambda_n<t}\phi_n^2(P)=(4\pi)^{-1}t^{\frac{1}{2}}+o(t^{\frac{1}{2}})$ as $t\rightarrow+\infty$. By integrating (iii) with proper care, it is shown that the number of eigenvalues between 0 and $t>0$ equals $(V/4\pi)t^{\frac{1}{2}}+o(t^{\frac{1}{2}})$ when $t\rightarrow+\infty$, where V is the area of domain V. Putting $t=\lambda_n$ one obtains $\lim_{n\rightarrow\infty}\lambda_n/n^2=(4\pi/V)^2$. *J. B. Dias.*

Popovici, Constantin. Prolongation of nonanalytical functions. Integrations by conditions on limit of linear equations with partial derivatives. An. Acad. Repub. Pop. Romåne. Ser. A. **2**, no. 32, 38 pp. (1949). (Romanian. Russian and English summaries)

The first part of the title refers to the unique determination of solutions of certain classes of partial differential equations by means of their values on certain sets. The author summarizes a number of such uniqueness theorems. He illustrates by means of hyperbolic equations of fourth order that some classical uniqueness theorems break down when solutions are admitted which do not possess the continuity properties required by the classical proofs. The paper is in part a survey of this field, on which the author has published many papers. *R. P. Boas, Jr.*

Functional Analysis, Ergodic Theory

★Miranda, Carlo. Problemi di esistenza in analisi funzionale. Scuola Normale Superiore, Pisa. Quaderni Matematici, no. 3. Litografia Tacchi, Pisa, 1950. 184 +ii pp.

The purpose of this book is to give an exposition of such parts of the theory of normed linear spaces as are useful in the establishment of existence theorems in various problems of analysis. It is not intended to be a complete treatise on the subject and not all proofs are given completely, the reader being frequently referred to the original papers for details. The theory is accompanied by a variety of applications to ordinary and partial differential equations and to integral equations, particularly nonlinear ones. The first chapter gives a brief introduction to some of the necessary preliminaries concerning normed linear spaces, and also examples of such spaces useful in the later applications. Chapter 2 opens with a brief exposition, without proofs, of F. Riesz's theory of completely continuous linear transformations in a Banach space. The rest of the chapter is devoted chiefly to the work of Caccioppoli [e.g., Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **11**, 794–799 (1930)], Hildebrandt and Graves [Trans. Amer. Math. Soc. **29**, 127–153 (1927)], and others on the method of successive approximation, local invertability and differentiability of transformations, and related topics. The third chapter introduces the notion of mapping degree, first in Euclidean n-space, then in Banach spaces, and leads up to some of the fixed point theorems of Schauder and Leray and others and their applications. There is also a rather extensive bibliography. *J. V. Wehausen* (Providence, R. I.).

Picone, Mauro, und Fichera, Gaetano. Neue funktionalanalytische Grundlagen für die Existenzprobleme und Lösungsmethoden von Systemen linearer partieller Differentialgleichungen. Monatsh. Math. **54**, 188–209 (1950).

This paper was given by Picone as an invited address before the Second Austrian Mathematical Congress in Innsbruck in 1949. He states in the introduction that he asked Fichera to write a certain part of the report. It is a crystalli-

zation in the form of an abstract theory of some of the methods used by the authors, and their associates at the National Institute for Applied Mathematics in Rome in the solution of problems involving differential and integro-differential equations [Picone, Ann. Sci. Univ. Jassy. Sect. I. **27**, 18–26 (1941); Fichera, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 527–532 (1947); Amerio, Amer. J. Math. **69**, 447–489 (1947); Ghizzetti, Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 131–168 (1946); these Rev. **8**, 464; **9**, 239, 37, 142]. The theory is formulated in terms of homogeneous distributive operators in a normed abstract function space which is linear with respect to the complex field. By specializing the space and the operators the general treatment is shown to fit problems involving systems of linear partial differential equations and integro-differential equations of the Volterra type.

H. P. Thielman (Ames, Iowa).

Lesky, Peter. Anwendung der Methode Picones auf ein Wärmeleitungsproblem. Monatsh. Math. **54**, 241–254 (1950).

This is an application of the method of Picone [see the preceding review] for the solution of linear partial differential equations. The temperature distribution over a square plate is determined under the hypothesis that the edges are kept at constant temperatures. The problem is chosen so that it can be solved by the classical method. The numerical results obtained by the classical method and by the new method are then compared. The author states that the agreement between these results is so good, that the practical usefulness of Picone's method has been demonstrated by this example.

H. P. Thielman (Ames, Iowa).

Fuglede, Bent. Linear operators in Hilbert space. Mat. Tidsskr. B. **1950**, 101–109 (1950). (Danish)

Expository paper on the spectral representation of linear operators.

Goodner, Dwight B. Projections in normed linear spaces. Trans. Amer. Math. Soc. **69**, 89–108 (1950).

A Banach space X has the property P_s ($s\geqq 1$) if and only if for every Banach space Y containing X there exists a projection of Y onto X of norm not greater than s. Akilov [Doklady Akad. Nauk SSSR (N.S.) **57**, 643–646 (1947); these Rev. **9**, 241] has shown that a B_1^+ space has P_s if its unit sphere has a least upper bound x_0 and $\|x_0\|\leqq s$. Goodner gives a new proof of this fact modeled after the Hahn-Banach theorem. The author also gives a proof of the equivalence between: (1) X has P_1 and the unit sphere in X has an extreme point; (2) X is equivalent to a space $C(H)$ of all real-valued continuous functions on some totally disconnected compact Hausdorff space H; and (3) X is equivalent to an abstract (M)-space with P_1. The author also shows that the conjugate space to an abstract (L)-space has P_1.

R. S. Phillips (Princeton, N. J.).

Yood, Bertram. Transitive systems of linear operators on a Banach space. Proc. Amer. Math. Soc. **1**, 509–511 (1950).

Démonstration du résultat suivant: Soit G un semi-groupe d'opérateurs linéaires d'un espace de Banach (de dimension infinie) X; on ne suppose du reste pas que les éléments de G soient définis partout, ou bornés; disons que G est transitif si, quels que soient les deux systèmes (x_i) et (y_i) ($1\leqq i\leqq n$, n fini arbitraire) d'éléments linéairement indépendants de X, il existe $T\epsilon G$ avec $Tx_i=y_i$; alors, pour que G soit transitif, il faut et il suffit que, pour tout sous-espace F de dimension finie de X, il existe $T\epsilon G$ qui se réduise à l'identité sur F et $\epsilon>0$ tel que, pour tout endomorphisme U de F vérifiant $\|U\|<\epsilon$, $T+U$ soit la restriction à F d'un élément de G (autrement dit, que les restrictions à F des éléments de G forment un voisinage de l'unité dans l'ensemble des endomorphismes de F). La condition est trivialement nécessaire; pour voir qu'elle est suffisante, soient (x_i) et (y_i) deux systèmes libres; X étant de dimension infinie, on peut les plonger dans un même sous-espace F de dimension p finie de telle sorte que, dans F, on puisse passer du système (x_i) au système (y_i) par une sustitution de déterminant 1; mais les restrictions à F des $T\epsilon G$ forment, dans l'algèbre des endomorphismes de F, un semi-groupe qui contient un voisinage de l'unité, donc contient le groupe engendré par ce voisinage, donc contient le groupe de toutes les matrices de déterminant 1 (car ce groupe est connexe), ce qui démontre le théorème.

R. Godement (Nancy).

Sz.-Nagy, Béla. Une caractérisation affine de l'ensemble des fonctions positives dans l'espace L^2. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 228–238 (1950).

Given a separable Hilbert space R with a positive cone P, the author determines necessary and sufficient conditions that R be isomorphic with a space L^2 [square summable functions relative to a distribution function on $(0, 1)$] such that the correspondents of P are precisely the functions $f(x)\geqq 0$ of L^2. The required conditions are: (A_1) If $u\epsilon P$, then $\lambda u\epsilon P$ for $\lambda\geqq 0$ and, if $u\neq 0$, then $\lambda u\epsilon P$ for $\lambda<0$; (A_2) if $u, v\epsilon P$, then $u+v\epsilon P$; (A_3) if $u_n\epsilon P$ and if $u_n\rightarrow u$, then $u\epsilon P$; (A_4) if $u_n, v_n\epsilon P$ and if $u_n+v_n\rightarrow 0$, then $u_n\rightarrow 0$; (A_5) if $f\epsilon R$, then there exist $u, v\epsilon P$ such that $f=u-v$; and (B) if $u_1, u_2, v_1, v_2\epsilon P$ and if $u_1+u_2=v_1+v_2$, then there exist $w_{11}, w_{12}, w_{21}, w_{22}\epsilon P$ such that $u_1=w_{11}+w_{12}$, $u_2=w_{21}+w_{22}$, $v_1=w_{11}+w_{21}$, $v_2=w_{12}+w_{22}$. The author gives a proof which makes use of a vector lattice representation for R.

R. S. Phillips (Princeton, N. J.).

Hewitt, Edwin. A note on normed algebras. Anais Acad. Brasil. Ci. **22**, 171–174 (1950).

Soit A une algèbre normée complète commutative sur le corps complexe, dépourvue d'élément unité. Soient K le corps complexe et K_{n11} l'algèbre obtenue en définissant la multiplication dans K par $xy=0$. L'auteur montre que si I est, dans A, un idéal maximal fermé, alors A/I est isomorphe à K ou bien à K_{n11}, conséquence triviale du théorème de Gelfand et Mazur. L'auteur donne ensuite un exemple du dernier cas en considérant l'ensemble des fonctions continues sur $[0, 1]$ muni du produit de composition de Volterra.

R. Godement (Nancy).

Ionescu Tulcea, C. T., et Marinescu, G. Théorie ergodique pour des classes d'opérations non complètement continues. Ann. of Math. (2) **52**, 140–147 (1950).

Let B be a linear manifold in the linear vector space E over the complex numbers. Let B and E be Banach spaces under the norms $\|x\|$, $|x|$, respectively. It is assumed that $x_n\epsilon B$, $\|x_n\|\leqq K$, and $|x_n-x|\rightarrow 0$ imply that $x\epsilon B$ and $\|x\|\leqq K$. Let $C(B, E)$ be the subclass of all the bounded linear transformations in the space B determined by the following three conditions: (i) $|T^nx|\leqq H|x|$, $x\epsilon B$, $n=1, 2, \cdots$, (ii) there exist positive constants R, r with $0<r<1$ such that $\|Tx\|\leqq r\|x\|+R|x|$, $x\epsilon B$, and (iii) TP is compact in E if P is bounded in B. Then for an operator T in this class

$C(B, E)$ it is shown that there are at most a finite number of characteristic numbers $c_1, \cdots, c_p$ of absolute value 1 and that $T^n = \sum_{i=1}^{p}(1/c_i^n)T_i + S^n$, $n=1, 2, \cdots$, where T_i is a projection in B with finite dimensional range, $T_iT_j=0$, $T_iS=0$, $i\neq j$, and $\|S^n\| \leqq M\delta^n$, $n=1, 2, \cdots$, with $\delta<1$. [Cf. Doeblin and Fortet, Bull. Soc. Math. France **65**, 132–148 (1937); and the authors, C. R. Acad. Sci. Paris **227**, 667–669 (1948); these Rev. **10**, 311.] *N. Dunford.*

Dixmier, Jacques. Les moyennes invariantes dans les semi-groupes et leurs applications. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 213–227 (1950).

The author considers invariant means over semi-groups. Let G be a topological semi-group and let C be the Banach space of real bounded continuous functions over G with the usual norm. A left invariant mean is a linear functional ϕ over C such that $\phi(f)\geqq 0$ if $f=f(x)\geqq 0$ $(f(x)\epsilon C, x\epsilon G)$, $\phi(1)=1$, and $\phi({}_sf)=\phi(f)$, where $s\epsilon G$ and ${}_sf(x)=f(sx)$. The fundamental theorem follows. In order that G possess a left invariant mean, it is necessary and sufficient that it have the property: If $f^1, \cdots, f^n\epsilon C$, $s_1, \cdots, s_n\epsilon G$, and a is a real number such that $f^1 - {}_{s_1}f^1 + \cdots + f^n - {}_{s_n}f^n \geqq a$, then $a\leqq 0$. It is shown that Abelian semi-groups have invariant means. Also, if the invariant sub-semi-group H possesses a left invariant mean and if the quotient semi-group G/H (discrete topology) does likewise, then so does G. Examples are given of semi-groups which do not possess left invariant means. This theory is then applied to the derivation of results due in modified form to various authors. If the group G possesses a right invariant mean, every strongly continuous bounded representation of G in a Hilbert space is equivalent to a unitary representation [B. Sz.-Nagy, Acta Univ. Szeged. Sect. Sci. Math. **11**, 152–157 (1947); these Rev. **9**, 191]. The structure of a general basis in Hilbert space is analysed [Lorch, Bull. Amer. Math. Soc. **45**, 564–569 (1939); these Rev. **1**, 58]. Proof is given of an ergodic theorem of Alaoglu and Birkhoff [Ann. of Math. (2) **41**, 293–309 (1940); these Rev. **1**, 339]. The existence of invariant measures is considered. In order to establish the last, a generalization of the Hahn-Banach theorem is proved. *E. R. Lorch.*

Aoki, K. On the billiard ball problem of space forms. Tensor **9**, 1–6 (1949). (Japanese)

After explanation of Birkhoff's results with respect to the billiard ball problem on a Euclidean plane [G. D. Birkhoff, Trans. Amer. Math. Soc. **18**, 199–300 (1917); Acta Math. **47**, 297–311 (1925); Dynamical Systems, Amer. Math. Soc. Colloquium Publ., v. 9, New York, 1927], the author remarks that these results also hold good on any Clifford-Klein space form. *A. Kawaguchi* (Sapporo).

Calculus of Variations

Tonelli, Leonida. Nuove ricerche su una speciale classe di problemi di calcolo delle variazioni. Rivista Mat. Univ. Parma **1**, 125–156 (1950).

In this paper, published after the author's death, detailed proofs are given of two theorems on the existence of a minimum for an integral of the form $I(y)=\int_{X_0}^{X_1}g(y, y')dx$. The integrand $g(y, y')$ is assumed to be defined for $Y_0\leqq y<Y_1$ for all y', and to have an infinite discontinuity at $y=Y_1$. In both theorems, admissible functions $y(x)$ are absolutely continuous for $X_0\leqq x\leqq X_1$, have $Y_0\leqq y(x)\leqq Y_1$, $y'(x)\geqq\psi(y(x))$ almost everywhere (where the function $\psi(y)$ is defined for $Y_0\leqq y\leqq Y_1$), and make $g(y(x), y'(x))$ integrable. Let $Y_0\leqq Y_i<Y_1$. In the first theorem it is shown, under suitable restrictions on g and ψ, that $I(y)$ has an absolute minimum in the (nonnull) class K consisting of all admissible $y(x)$ with $y(X_0)=Y_i$, and that a minimizing function is of class C'. It is also shown that no function in K has $y(x)=Y_1$ at any point. In the second theorem, under somewhat different restrictions on g, it is shown that $I(y)$ has an absolute minimum in the (nonnull) class K_1 consisting of all admissible $y(x)$ such that $y(X_0)=Y_1$ and $y(x)<Y_1$ for $x>X_0$, and that a minimizing function has a continuous derivative for $X_0<x\leqq X_1$. The first of these theorems was treated by the author in a previous memoir [Ann. Scuola Norm. Super. Pisa (2) **10**, 167–189 (1941) = Ist. Naz. Appl. Calcolo (2), no. **123**; these Rev. **6**, 180] using a different method of proof. The second theorem was announced previously [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 247–250 (1946); these Rev. **9**, 148]. *L. M. Graves.*

Giannopoulos, Alex. I. Solution of a problem of the calculus of variations. Bull. Soc. Math. Grèce **24**, 129–132 (1949). (Greek)

The author tries to establish conditions under which Euler's, Legendre's, and Jacobi's necessary conditions together are sufficient for a strong minimum, of

$$J=\int_a^b f(x, y, y')dx.$$

The argument used is invalid because he erroneously assumes that $\delta J=0$ and $\delta^2 J>0$ are sufficient to insure a minimum. *J. Dugundji* (Los Angeles, Calif.).

Magenes, Enrico. Sui teoremi di Tonelli per la semicontinuità nei problemi di Mayer e di Lagrange. Ann. Scuola Norm. Super. Pisa (2) **15** (1946), 113–125 (1950).

The author gives a detailed proof of the following theorem of Tonelli [see Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **24**, 399–404 (1937)]. Let $x=x(t)$, $y=y(t)$ be an absolutely continuous parametric representation of a curve C, and let $u(t, C, a)$ denote the solution of the equation $u=a+\int_{t_1}^{t}F(x, y, x', y', u)\,dt$ when it exists. Suppose F satisfies the conditions: (A) there exists a fixed R such that $[F(x, y, x', y', u_1)-F(x, y, x', y', u_2)]/(u_1-u_2)\leqq R$; (B) $E(x, y, x', y', \bar{x}', \bar{y}', u)\geqq 0$ everywhere; (C) for each fixed (x, y, x', y', u), $E\neq 0$ in $(\bar{x}', \bar{y}')$. Then $u(t, C, a)$ is lower semicontinuous at each (C, a) where it is defined, uniformly in t. A corresponding theorem (with appropriate modification of hypothesis (A)) is proved for nonparametric problems. *L. M. Graves* (Chicago, Ill.).

Magenes, Enrico. Intorno agli integrali di Fubini-Tonelli. I. Condizioni sufficienti per la semicontinuità. Ann. Scuola Norm. Super. Pisa (3) **2** (1948), 1–38 (1950).

Magenes, Enrico. Intorno agli integrali di Fubini-Tonelli. II. Teoremi di esistenza dell'estremo. Ann. Scuola Norm. Super. Pisa (3) **3** (1949), 95–131 (1950).

These two papers are concerned with the "Fubini-Tonelli" integrals:

$$I(y_1, y_2)=\int_a^b\int_c^d f(x, z, y_1(x), y_2(z), y_1'(x), y_2'(z))\,dx\,dz,$$

$$I(y)=\int_a^b\int_c^d f(x, z, y(x), y(z), y'(x), y'(z))\,dx\,dz.$$

In the first paper the author applies the direct method of Tonelli to determine conditions sufficient for the lower semi-continuity of these integrals. For the major portion of the work attention is restricted to the class of curves C: $y=y_1(x)$, $y=y_2(z)$, $a\leqq x\leqq b$, $c\leqq z\leqq d$, which are "ordinary" in the sense that: (i) $(x, y_1(x))$, $a\leqq x\leqq b$, is in a region A_1 of the (x, y_1)-plane, and $(z, y_2(z))$, $c\leqq z\leqq d$, is in a region A_2 of the (z, y_2)-plane, where $f(x, z, y_1, y_2, y_1', y_2')$ is continuous and has continuous first and second order derivatives with respect to y_1', y_2' for (x, z, y_1, y_2) in $A=A_1\times A_2$ and y_1', y_2' arbitrary; (ii) $y_1(x)$, $a\leqq x\leqq b$, and $y_2(z)$, $c\leqq z\leqq d$, are absolutely continuous functions; (iii) $I(y_1, y_2)$ exists as a finite Lebesgue integral. The hypotheses of the individual theorems are too complicated to state specifically; in particular, the results obtained extend those of S. Faedo [Ann. Mat. Pura Appl. (4) **23**, 69–121 (1944); these Rev. **7**, 525].

In the second paper the author gives for each of the above integrals various theorems on the existence of an absolute minimum in a complete class of ordinary curves, both in the case of the region A bounded and in the case of A unbounded. Again, statements of individual theorems are too complicated to give here. All existence theorems established for $I(y_1, y_2)$ are for a complete class K_η of ordinary curves C such that there exists a positive constant η satisfying $d-c\geqq\eta$, $b-a\geqq\eta$ for every curve C of K_η; for $I(y)$ the author obtains a further group of existence theorems involving an arbitrary complete class of ordinary curves.

W. T. Reid (Evanston, Ill.).

Faedo, Sandro. Sulle condizioni di Legendre e di Weierstrass per gli integrali di Fubini-Tonelli. Ann. Scuola Norm. Super. Pisa (2) **15** (1946), 127–135 (1950).

For the functional

$$I(y_1, y_2)=\int_a^b\int_c^d f(x, z, y_1(x), y_2(z), y_1'(x), y_2'(z))\,dz\,dx$$

the author gives an example of an integrand,

$$f=\exp(y_1'^2y_2'^2)-y_1'^2,$$

and a curve, $\bar{C}$: $\bar{y}_1=0$, $\bar{y}_2=\frac{3}{4}z^{4/3}$, at which $I(y_1, y_2)$ is lower semi-continuous, and the Weierstrass condition

$$\int_c^d E_1[x, z, \bar{y}_1(x), \bar{y}_2(z), \bar{y}_2'(z); \bar{y}_1'(x), y_1']\,dz\geqq 0,$$

$$\int_a^b E_2[x, z, \bar{y}_1(x), \bar{y}_2(z), \bar{y}_1'(x); \bar{y}_2'(z), y_2']\,dx\geqq 0$$

holds, but the Legendre condition

$$\int_c^d f_{y_1'y_1'}[x, z, \bar{y}_1(x), \bar{y}_2(z), \bar{y}_1'(x), \bar{y}_2'(z)]\,dz\geqq 0,$$

$$\int_a^b f_{y_2'y_2'}[x, z, \bar{y}_1(x), \bar{y}_2(z), \bar{y}_1'(x), \bar{y}_2'(z)]\,dx\geqq 0$$

fails. *L. M. Graves* (Chicago, Ill.).

Sigalov, A. G. Regular double integrals of the calculus of variations in nonparametric form. Doklady Akad. Nauk SSSR (N.S.) **73**, 891–894 (1950). (Russian)

A function f defined on the closure of a bounded region D of the (x, y) plane belongs to the class L_α, $\alpha\geqq 1$, if (1) on all segments lying in D and parallel to the y-axis f is an absolutely continuous function of x, and vice versa, (2) $(p^2+q^2)^{\frac{1}{2}\alpha}$ is summable over D, (3) f is continuous on $\bar{D}$. Let H consist of all $P=(x, y, z)$ with $(x, y)\epsilon D$ and z real. The principal theorem is the following. Let D be a bounded region such that the components of its boundary have diameters bounded from 0. Let $F(P, p, q)$ be defined and continuous, together with its derivatives F_p, F_q, for all P in H and all real p, q. Assume further (1) $\mathcal{E}(P, p, q, \bar{p}, \bar{q})\geqq 0$ for all $P\epsilon H$ and all $p, q, \bar{p}, \bar{q}$; (2) there exist positive constants L, m, M, and $\alpha>1$ such that (2a) $F(P, p, q)\geqq m(1+p^2+q^2)^{\frac{1}{2}\alpha}$ for $p^2+q^2\geqq L$, $P\epsilon H$; (2b) $|F(P, p, q)|\leqq M$ for $p^2+q^2\leqq L$, $P\epsilon H$; (3) to each bounded closed subset H_1 of H corresponds a convex function $\varphi(p, q)$ and positive L_1, L_2, L_3 such that

$$L_1\varphi(p, q)\leqq F(P, p, q)\leqq L_2\varphi(p, q)$$

for $P\epsilon H_1$ and $p^2+q^2\geqq L_3$.

Let ψ be defined and continuous on the boundary of D. A function f on $\bar{D}$ is "admissible" if it is in L_α, coincides with ψ on the boundary of D, and $\iint_D F(x, y, f, p, q)\,dx\,dy<\infty$. Then if the class of admissible functions is not empty, it contains a minimizing function for $\iint_D F(x, y, f, p, q)\,dx\,dy$.

This theorem is considerably stronger than others in the literature, in that α is assumed only >1 and nevertheless the minimizing function retains strong continuity properties. A useful auxiliary is $Q\{f, G\}_\alpha$, the ratio of the integral of $(p^2+q^2)^{\frac{1}{2}\alpha}$ over G to the square of the length of the curve $(x, y, f(x, y))$ with (x, y) on the boundary of G. If this is small, the oscillation of f on $\bar{G}$ cannot much exceed its oscillation on the boundary of G; if it is large, f can be smoothed without increasing the integral of F. The principal lemmas needed in the proof are stated, but the proofs are either condensed or omitted. *E. J. McShane.*

McShane, E. J. A metric in the space of generalized curves. Ann. of Math. (2) **52**, 328–349 (1950).

In the topology of generalized curves introduced by the reviewer, the relation $C_n\to C$ means that for every admissible variational integrand $F(y, \dot{y})$, $I(C_n, F)\to I(C, F)$, where $I(C, F)$ denotes the curvilinear integral of F over the curve C. This topology induces a change in the notion of identity for classical curves; C_1 and C_2 are identified whenever for every admissible F we have $I(C_1, F)=I(C_2, F)$. (Without this identification C_2 would be a second solution to every variational problem having the solution C_1.) The author proposes here a modification of the topology so as not to disturb the classical notion of identity (at the expense, however, of unicity of solutions). His object is to prepare for a suitable generalization of the theory of Morse. The modification does not invalidate metrizability, compactness, or the existence theorems. It consists essentially in using in place of $I(C, F)$ the indefinite integral $I(\gamma, F)$, where γ is a variable arc of C. The final part of the paper treats nonparametric generalized curves, not as a distinct species, but as a special case of parametric generalized curves, and the author establishes a general existence theorem for the problem of Bolza in nonparametric form. *L. C. Young.*

Theory of Probability

Insolera, F. Considerazioni sulla "frequenza totale" e sulle "frequenze composte." Giorn. Mat. Finanz. (3) **8**, 43–58 (1950).

Hammersley, J. M. The distribution of distance in a hypersphere. Ann. Math. Statistics **21**, 447–452 (1950).

Let R be a fixed point in a vector space T_1 of n dimensions and let $S_n(a)$ be the n-dimensional hypersphere in T_1 with

center R and radius a. Let A and B be any two points chosen at random in $S_n(a)$, the distributions of A and B being independent and uniform over the interior of $S_n(a)$. Denote the distance AB by r; and let $\lambda=r/2a$, so that λ may take any value in the interval $0\leqq\lambda\leqq 1$. The author derives the frequency function of λ and proves for large n that the asymptotic distributions of λ and r are normal.

L. A. Aroian (Culver City, Calif.).

Takano, Kinsaku. A note on the concentration functions. Kōdai Math. Sem. Rep. **1950**, 13 (1950).

The distance $\rho(F, G)$ between two distribution functions is defined by P. Lévy as the maximum distance between the corresponding curves measured in the direction of the line $x+y=1$. The concentration function is defined by $Q_F(l)=\sup\{F(x+l+0)-F(x-0)\}$ for $l>0$, and by $Q_F(l)=0$ for $l\leqq 0$. It is a distribution function, and the author remarks that $\rho(Q_F, Q_G)\leqq 2\rho(F, G)$. *W. Feller.*

Obrechkoff, N. Sur quelques lois asymptotiques de probabilités et sur les solutions bornées de quelques équations intégrales singulières et des équations linéaires à un nombre infini des inconnues. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **43**, 269–349 (1947). (Bulgarian. French summary)

[Volume number misprinted **42** on title page.] Suppose that $p(x)\epsilon L(-\infty, \infty)$, that $\|p\|=1$ (in the L-norm) and that for the n-fold convolution of $p(x)$ with itself one has $\|p_n(x)-p_n(x-h)\|\rightarrow 0$ for each fixed h. Simple estimates then show that the integral equation (1) $f(x)=\int_{-\infty}^{+\infty}p(t)f(x+t)dt$ can have no bounded nonconstant solution. Similarly, if (2) $\phi_n(x)=\int_{-\infty}^{+\infty}p(t)\phi_{n+1}(x+t)dt$ and the sequence ϕ_n is uniformly bounded, then each $\phi_n(x)$ equals a constant. Analogous theorems are proved for the interval $(0, \infty)$ and for sequences. Putting, in particular, $p(x)=e^{-x}x^{\alpha-1}/\Gamma(\alpha)$ (where $x>0$) and $f(t)=\phi(t)e^t$, the equation (1) involves a fractional integral and permits one to conclude that, if $|f^{(n)}(x)|<Me^{-x}$ for $x>a$ and all n, then $f(x)=M_1e^{-x}$. This result is due to Tagamlitzki [same Annuaire. Livre 1. **42**, 239–256 (1946); these Rev. **9**, 14]. The author gives various generalizations. Next the author derives various results of the same general type by making assumptions in the bilateral Laplace transform of $p(x)$ and using the results of the paper reviewed below. Finally the results are generalized to functions of two variables. *W. Feller* (Princeton, N. J.).

Obrechkoff, N. Sur quelques lois asymptotiques de probabilités. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. (Math. Phys.) **44**, 201–233 (1948). (Bulgarian. French summary)

Let $p(x)$ be a density function and denote its bilateral Laplace transform by $g(s)$. Suppose that $g(s)$ is analytic in a strip $-\alpha<\Re s<\beta$ and that therein $g(s)=O(|s|^{-k})$ as $|s|\rightarrow\infty$ (here α, β, k are positive). Put $a=-g'(0)$, and denote by δ the root of $\omega+g'(s)/g(s)=0$. By means of estimates of the type used in the method of steepest descent the author proves the following corollary to the central limit theorem: For ω sufficiently near to a and $\omega<a$, one has the following asymptotic relation for the nth convolution $p_n(x)$ of $p(x)$ with itself.

$$\int_{-\infty}^{n\omega}p_n(x)dx\sim e^{n\omega\delta}g^n(\delta)\delta^{-1}(2\pi n)^{-\frac{1}{2}}\{g''(\delta)/g(\delta)-\omega^2\}^{-\frac{1}{2}};$$

for $\omega>a$ one gets a similar estimate for the integral over $(n\omega, \infty)$.

A similar theorem is proved for arithmetical distributions. In particular, for the binomial distribution one gets for ω sufficiently near to np and $\omega<np$

$$\sum_{\nu=0}^{[n\omega]}\binom{n}{\nu}p^\nu q^{n-\nu}\sim(p/\omega\gamma^{\omega-1})^n(1-\gamma)^{-1}\{2\pi n(\omega-\omega^2)\}^{-\frac{1}{2}},$$

where $\gamma=q\omega/p(1-\omega)$. As an application the author proves that certain recurrently defined sequences of functions can converge only to a constant. This paper is related to another one by the author [see the preceding review].

W. Feller (Princeton, N. J.).

Ledermann, Walter. On the asymptotic probability distribution for certain Markoff processes. Proc. Cambridge Philos. Soc. **46**, 581–594 (1950).

The author considers a Markov chain with a continuous parameter. The transition probability matrix can be put in the form $P(t)=e^{-tA}$, where A is a constant matrix with vanishing row sums whose elements not on the main diagonal are $\leqq 0$. The asymptotic properties of the chain when $t\rightarrow\infty$ are determined by those of $P(t)$. The author finds the asymptotic properties of $P(t)$ (for example, he shows that $\lim_{t\rightarrow\infty}P(t)$ exists) by algebraic methods. As he remarks, his final results are not new. [See, for example, Doeblin, Bull. Sci. Math. (2) **62**, 21–32 (1938); **64**, 35–37 (1940); these Rev. **1**, 343.] *J. L. Doob* (Urbana, Ill.).

Lévy, Paul. Éléments de la théorie des processus à la fois stationnaires et de Markoff, dans le cas d'un système ayant une infinité dénombrable d'états possibles. C. R. Acad. Sci. Paris **231**, 467–468 (1950).

If the transition probabilities $P_{jk}(t)$ are continuous, the derivatives $P'_{jk}(0)$ can be shown to exist. Put $\lambda_j=-P'_{jj}(0)$. The succession of states through which the system passes can be studied without regard to t as a simple Markov chain with constant transition probabilities p_{jk}. The total time it takes the system to pass through a given sequence $(H_{j_1}, \cdots, H_{j_n})$ is a random variable of the form $\sum_{k=1}^{n}X_{j_k}/\lambda_{j_k}$, where the X_{j_k} are mutually independent random variables with the common distribution $1-e^{-t}$. The author puts $S=\sum 1/\lambda_{j_k}$ and says that we have the finite or infinite case according as S diverges or converges. [Actually, the convergence of the series can have any probability between 0 and 1. The two cases which the author has in mind were studied, also for nondenumerable chains, by the reviewer [Trans. Amer. Math. Soc. **48**, 488–515 (1940); these Rev. **2**, 101] who gave necessary and sufficient conditions for their occurrence.] In the finite case the number of jumps during any finite time interval is finite and the chain C together with the λ_j completely determines the process. In the infinite case the jumps can accumulate to a finite instant t_0, and one has the freedom of prescribing the state into which the system is to pass at such a moment, so that in this case several processes can have the same infinitesimal transition probabilities [this phenomenon has been discussed by Doob, Trans. Amer. Math. Soc. **58**, 455–473 (1945); these Rev. **7**, 210]. *W. Feller.*

Popoff, Kyrille. Observations sur la théorie des probabilités en chaîne de Markoff. Cas d'une suite continue d'épreuves. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **38**, 319–330 (1942). (Bulgarian. French summary)

Kolmogoroff ayant ramené l'étude du cas d'une suite continue d'épreuves dans un système, n'admettant qu'un nombre

fini d'états possibles $E_1, E_2, \cdots, E_r$, à l'étude d'un système d'équations linéaires $dx_{jk}/dt=\sum_{i=1}^{r} u_{ki}x_{ji}$, $i, j=1, \cdots, r$, dont les coefficients satisfont aux conditions (*) $\sum_{i=1}^{r} u_{ki}(t)=0$, $u_{ki}(t)\geqq 0$ for $i\neq k$, $u_{ii}(t)\leqq 0$, nous avons étudié le caractère analytique des intégrales de ces équations dans les cas suivants: (a) les coefficients u_{ki} sont des constantes; (b) les coefficients $u_{ki}(t)$ sont des fonctions de t, tendant vers des limites bien déterminées, lorsque t tend vers $+\infty$; (c) les coefficients $u_{ki}(t)$ sont des fonctions quelconques de t, satisfaisant aux conditions (*). *Author's summary.*

Yosida, Kôsaku. An extension of Fokker-Planck's equation. Proc. Japan Acad. 25, no. 9, 1–3 (1949).

The transition probabilities of each Markov process determine a semi-group of transformations on the initial distribution. For processes of the diffusion type the infinitesimal generator is given [at least in some typical cases] by the right side of the Fokker-Planck equation. For processes of the "purely discontinuous" type one has a bounded operator, and the combination of the two leads to integro-differential equations in which the right side of the usual Fokker-Planck equation is augmented by a bounded integral operator [Feller, Math. Ann. 113, 113–160 (1936)]. The stable processes are characterized by an unbounded operator which is determined by a certain monotonic function $\Omega(x)$. The author shows how the integro-differential equation mentioned above can be generalized using $\Omega(x)$ as kernel.

W. Feller (Princeton, N. J.).

Giltay, J. Static and transient statistics in telephone-traffic problems. Appl. Sci. Research B. 1, 413–419 (1950).

The solutions of Kolmogorov's differential equations for a Markov process with finitely many states are linear combinations of exponentials. It has frequently been remarked that the real parts of the nonvanishing exponents are negative, so that the solutions approach limits as $t\to\infty$. The author proves this anew. In applications it is customary to write down only the right sides of the equations and to calculate from them the "steady state." The author considers that he has now justified this procedure.

W. Feller (Princeton, N. J.).

Borovickiĭ, S. I. On fluctuations in a linear system with periodically varying parameters. Doklady Akad. Nauk SSSR (N.S.) 74, 233–236 (1950). (Russian)

The deterministic equation of motion $\dot{x}=a(t)x+b(t)$ is replaced by the diffusion equation $w_t=-\{(ax+b)w\}_x+c(t)w_{xx}$ whose fundamental solution can be written down in terms of the normal distribution. The author supposes that the coefficients a, b, c are periodic functions. The diffusion equation then determines a process for $-\infty<t<\infty$ and the author calculates the mean, dispersion, Fourier transform, and spectrum of the sample function. *W. Feller.*

Blanc-Lapierre, André. Quelques modèles statistiques utiles pour l'étude du bruit de fond. C. R. Acad. Sci. Paris 231, 566–567 (1950).

Let $\{t_j\}$ be a sequence of points in a Poisson distribution with density $\rho(t)$. (1) If $x(t)=\sum_j R(t-t_j)$, and if $X(t)=x(t)-E\{x(t)\}$, it is stated that the spectrum of $E\{|X(t)|\}$ is the same as if the Poisson distribution had a constant density with value equal to the average value of $\rho(t)$. (2) Define $Z(t)$ to be 1 [-1] with probability $\frac{1}{2}$ [$\frac{1}{2}$] when $t=0$, and otherwise to be $Z(0)(-1)^n$, where n is the number of t_j's between 0 and t. The author gives the covariance function of the $Z(t)$ stochastic process, and the (average) spectrum, under various hypotheses on $\rho(t)$.

J. L. Doob (Urbana, Ill.).

Mathematical Statistics

***Neyman, J. First Course in Probability and Statistics.** Henry Holt and Co., New York, N. Y., 1950. ix+350 pp. $3.50.

Ce livre est un livre de classe destiné aux débutants, et parmi eux à ceux qui n'ont d'autre instruction mathématique que celle de l'enseignement secondaire; il ne développe donc aucune méthode avancée; par des exemples simples mais soigneusement discutés même dans leur aspect pratique, il cherche à faire comprendre ce qu'est la probabilité, ce qu'est le calcul de probabilités et quels problèmes relèvent de lui, ce qu'est la statistique et quels problèmes en relèvent, ce que peut signifier "tester une hypothèse statistique," etc. L'auteur limite ses exemples à des cas où une nombre fini d'éventualités seulement est à envisager, mais, sauf pour la probabilité elle-même, il s'efforce de poser des définitions générales. En plus des exemples, des exercises proposés permettront au lecteur de s'assurer qu'il a bien compris ces définitions. L'auteur a surtout voulu amener le lecteur au problème des tests d'hypothèses exposé au chapitre V, et les chapitres précédents (élements sur les probabilités, variables aléatoires, etc.) sont surtout une nécessaire préparation au chapitre V.

Compte tenu du but poursuivi et des difficultés qu'il implique, l'auteur a parfaitement réussi, principalement en n'omettant jamais de regarder en detail si le problème mathématique traité exprime correctement le problème concret initial; la seule critique que je lui adresserais concerne le chapitre I; celui-ci est une introduction, pour convaincre le lecteur de lire la suite, où sont presentées non seulement la notion de probabilité, mais aussi celle de "comportement inductif"; la réduction du choix d'une conduite à une schéma mathématique est une problème si difficile que la science ne l'a abordé que récemment et je crains qu'il ne trouble de très jeunes étudiants autant qu'il les intéressera: il aurait peut être mieux valu n'en parler qu'au début du chapitre V.

Voici une sommaire de chaque chapitre; je viens de signaler le contenu du chapitre I; le chapitre II définit la probabilité (rapport du nombre des cas favorables au nombre des cas possibles, pour une collection finie donnée de cas possibles), la probabilité conditionnelle, l'indépendance stochastique, et donne les formules élementaires de l'analyse combinatoire; la principale illustration est un problème de "risques concurrents." Le chapitre III est consacré à la théorie chromosomique de l'hérédité; l'auteur en présente un exposé assez complet (incluses les questions de linkage), qu'il traduit en axiomes convenables à un traitement probabiliste; la considération de générations successives lui permet de présenter des chaines de Markoff, et même des cas de tendance vers une distribution stationnaire; l'objet du chapitre est surtout de fournir une illustration intéressante du précédent, mais tel quel il pourrait suffire à beaucoup de biologistes.

Le chapitre IV définit les variables aléatoires et les fonctions de répartition, étudie les distributions binomiales, hypergéométriques, et obtient les distributions normales et de Poisson, connue approximation de la binomiale. Le chapitre V étudie les tests d'hypothèses; une terminologie

sûre est d'abord établie (hypothèses statistiques, hypothèses simples ou complexes, ensembles des hypothèses admissibles, test, erreurs de première et de seconde espèce, région critique, degré de sécurité, puissance d'un test); le problème et la terminologie précédente sont illustrées par 2 très bons exemples (dépistage de la tuberculose, problème de la "dame goûtant du thé"); dans le cas du choix entre 2 hypothèses simples seulement, les "meilleures régions critiques" sont définies et les problèmes résultant abordés; l'extension possible à des hypothèses complexes, grâce à une "région critique uniformément la meilleure" est envisagée (avec application au contrôle de fabrication) et l'auteur termine par la considération du "principe du λ." *R. Fortet.*

Krishna Iyer, P. V. Further contributions to the theory of probability distributions of points on a line. I. J. Indian Soc. Agric. Statistics 2, 141–160 (1950).

This paper continues previous work of the author [same J. 1, 173–195 (1948); these Rev. **11**, 446]. The author obtains the moments and difference equations satisfied by the moment generating functions of the distributions of certain of his previously considered statistics in "free" sampling. *J. Wolfowitz* (New York, N. Y.).

Smirnov, N. V. On the construction of confidence regions for the density of distribution of random variables. Doklady Akad. Nauk SSSR (N.S.) **74**, 189–191 (1950). (Russian)

Let $\xi_1, \cdots, \xi_n$ be mutually independent random variables with a common distribution function of density f. Suppose that f is continuous and positive in the interval $[a, b]$, that $\int_a^b f(x)dx<1$, and suppose that the integers s, n satisfy the relation $s^3(\log s)^3=O(n)$, where $s\rightarrow\infty$ with n. Divide $[a, b]$ into s intervals $\Delta_1, \cdots, \Delta_s$ of equal length h. If $x\epsilon\Delta_k$, let $f_n^*(x)$ be the number of ξ_j's in Δ_k, divided by nh, and let $\bar{f}(x)=h^{-1}\int_{\Delta_k}f(y)\,dy$. Then for all λ,

$$\Pr\left\{\max_{a\leq x\leq b}\frac{f_n^*(x)-\bar{f}(x)}{(\bar{f}(x))^{\frac{1}{2}}}\leqq\frac{l_s+\lambda/l_s}{(nh)^{\frac{1}{2}}}\right\}=e^{-2e^{-\lambda}}+O[(\log s)^{-\frac{1}{2}}].$$

Here l_s is the root of the equation $\frac{1}{2}-(2\pi)^{-\frac{1}{2}}\int_0^z e^{-\frac{1}{2}y^2}dy=1/s$. Let ϕ_n^* be the function whose graph is the polygonal line joining the points $(x_k, f_n^*(x_k))$, where x_k is the midpoint of Δ_k. Then if in addition to the other conditions $n\log s=o(s^5)$, and if f has a bounded second derivative in $[a, b]$,

$$\lim_{n\rightarrow\infty}\Pr\left\{\max_{a\leq x\leq b}\frac{\phi_n^*(x)-f(x)}{(f(x))^{\frac{1}{2}}}\leqq\frac{l_s+\lambda/l_s}{(nh)^{\frac{1}{2}}}\right\}=e^{-2e^{-\lambda}}.$$

J. L. Doob (Urbana, Ill.).

Birnbaum, Z. W., and Chapman, D. G. On optimum selections from multinormal populations. Ann. Math. Statistics **21**, 443–447 (1950).

Let a performance score X have the conditional distribution

$$f(X\,|\,Y_1, \cdots, Y_n)=(2\pi)^{-\frac{1}{2}}\sigma^{-1}\exp\left[-(X-\textstyle\sum_{i=1}^n\rho_iY_i)^2/2\sigma^2\right],$$

where $Y_1, \cdots, Y_n$ are the results of n tests which are to be used for selecting a subpopulation π^* from an initial population π such that the ϵ-quantile of X in π^* is greater than or equal to some given number. It is shown that a selection satisfying this condition and retaining as large a portion of π as possible is given by a linear truncation $\sum_{i=1}^n\rho_iY_i\geqq$a constant. The problem of insuring that EX in π^* is greater than or equal to a given number is solved in the same way. *G. E. Noether* (New York, N. Y.).

Richter, Hans. Über die Teststärke des Fisherschen Testes. Z. Angew. Math. Mech. **30**, 197–203 (1950). (German. English, French, and Russian summaries)

Let p_i $(i=A, B)$ be the probability that an individual subjected to treatment i react favorably; let m individuals be subjected to treatment A and n to B. It is shown that Fisher's test of the hypothesis $p_A=p_B$ based on holding all marginal totals of the 2×2 table fixed has a power function depending only on $\xi=\log p_B(1-p_A)/[p_A(1-p_B)]$ in addition to the marginal totals and level of significance. These results are related to those of E. S. Pearson and M. Merrington [Biometrika **35**, 331–345 (1948); these Rev. **10**, 388]. *T. W. Anderson* (New York, N. Y.).

Votaw, D. F., Jr., Kimball, A. W., and Rafferty, J. A. Compound symmetry tests in the multivariate analysis of medical experiments. Biometrics **6**, 259–281 (1950).

The authors treat some special cases of tests of compound symmetry considered by Votaw [Ann. Math. Statistics **19**, 447–473 (1948); these Rev. **10**, 387]. An application to a medical experiment is given with computational methods. Some analysis of variance tests are considered. Distributions of criteria are given in several particular cases. *T. W. Anderson* (New York, N. Y.).

Grant, David A. Statistical theory and research design. Annual Review of Psychology **1**, 277–296 (1950).

An expository treatment of the subject of the title together with a rather extensive bibliography.

✶Odier, Marcel. Les moments infinitésimaux et le problème du risque dans les caisses de pensions autonomes. Thesis, University of Zürich, 1945. 55 pp.

The moments and cumulants of a discrete random process in the time interval $(t, t+dt)$ are equivalent and, provided the process is time-homogeneous [a fact not mentioned by the author] they may be added to derive the cumulants, and thus the moments, for the time interval $(0, t)$. The distribution function of the process may then be expressed as a Gram-Charlier Type A series. An application of this development (which is nonrigorous) is made to the problem of the stability of a pension fund comprising 1,000 active members. *H. L. Seal* (New York, N. Y.).

TOPOLOGY

Arens, Richard. Note on convergence in topology. Math. Mag. **23**, 229–234 (1950).

It is well known that there are close connections between spaces topologized by a closure operator and spaces in which a notion of convergence is introduced by the use of directed systems. In the note under review, the author lists a number of axioms for convergence, defines convergence when a topology is given, and defines a topology when a notion of convergence is given. Necessary and sufficient conditions are given for the transition, convergence→topology→convergence, to reproduce the original notion of convergence. Several examples are given; in particular, the rôle of con-

vergence-in-measure in integration theory is explained by its equivalence with a certain closure operator.
E. Hewitt (Seattle, Wash.).

Nagata, Jun-iti. On the uniform topology of bicompactifications. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 1, 28–38 (1950).

This paper characterizes that uniform structure among the totally bounded uniform structures for which the completion is the Čech compactification as the uniform structure in which: If f and g are continuous real-valued functions on X without common zeros, then there exists a uniformly continuous h such that $h(x)=0$, respectively 1, when $f(x)$, respectively $g(x)$, vanishes. A similar characterization is made for Wallman's compactification. Also, spaces are investigated in which disjoint closed sets always have disjoint ϵ-neighborhoods for some ϵ in the uniform structure. In the case of a metric such spaces are complete and even compact when there are no isolated points. *R. Arens.*

Whitehead, J. H. C. Teoria della dimensione. Boll. Un. Mat. Ital. (3) 5, 156–164 (1950).
Expository lecture.

***van der Waerden, B. L. The Jordan-Brouwer theorem.** Zeven voordrachten over topologie. [Seven Lectures on Topology]. Centrumreeks, no. 1. Math. Centrum Amsterdam, pp. 80–84. J. Noorduijn en Zoon, Gorinchem, 1950. 6 florins.

An exposition embodying a proof of the Jordan curve theorem, based on the methods used by J. W. Alexander.
R. L. Wilder (Ann Arbor, Mich.).

Harrold, O. G., Jr. Euclidean domains with uniformly Abelian local fundamental groups. II. Duke Math. J. 17, 269–272 (1950).

Let C be a topological k-cell in the n-sphere S^n and let T be a topological r-cell contained in C. If S^n-C is unalg [terminology of part I, Trans. Amer. Math. Soc. 67, 120–129 (1949); these Rev. 11, 381] S^n-T is simply connected.
R. H. Fox (Princeton, N. J.).

Leray, Jean. L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue. J. Math. Pures Appl. (9) 29, 1–80, 81–139 (1950).

This is a detailed account of the author's cohomology theory of a topological space and of a continuous mapping of one space into another, based on lectures given at the Collège de France in 1947–48, 1949–50. Although the author prefers to use the term homology for what is usually known as cohomology, we shall follow the customary usage to avoid confusion. A résumé of the main results has already appeared [Topologie algèbrique, pp. 61–82, Colloques Internationaux . . . , no. 12, Centre de la Recherche Scientifique, Paris 1949; these Rev. 11, 677].

This paper is divided into three chapters. Chapter I gives the algebraic preliminaries. A filtered ring is a ring $\mathfrak{A}$ together with a mapping (filtration) f of $\mathfrak{A}$ into integers augmented by $+\infty$ such that $f(a-a')\geqq\min[f(a), f(a')]$, $f(aa')\geqq f(a)+f(a')$; $f(0)=+\infty$, $a, a'\epsilon\mathfrak{A}$. A graded ring is a particular example of a filtered ring, if we take as the filtration of a nonzero element its degree. A differential ring is a ring $\mathfrak{A}$ together with an automorphism α and a linear mapping δ of $\mathfrak{A}$ into itself such that $\delta^2a=0$, $\alpha\delta a+\delta\alpha a=0$, $\delta(aa')=\delta a\cdot a'+\alpha a\cdot\delta a'$, $a, a'\epsilon\mathfrak{A}$. To a differential filtered ring the author defines its spectral cohomology ring, which is a sequence of graded differential rings H_r depending on an integer r such that the differentiation of H_r is homogeneous of degree r and H_{r+1} is the cohomology ring of H_r. Various properties of the spectral cohomology ring are established. The rest of the chapter deals with the tensor product of differential filtered rings and its spectral cohomology ring.

Chapter II studies the spectral cohomology ring and the filtered cohomology ring relative to a differential filtered "faisceau" of a locally compact space and of a continuous mapping of one such space into another. A faisceau B on a space X is defined by giving: (1) a ring $B(F)$ for each closed subset F of X; (2) a homomorphism $B(F)\rightarrow B(F_1)$ when $F_1\subset F$, such that certain conditions are satisfied. Examples of faisceaux include the rings of cochains in the closed subsets of X and the coefficient rings. Special but important faisceaux can be defined by mapping a differential ring $\mathfrak{K}$ into the family of closed subsets of X, with certain conditions, the set $S(k)$, $k\epsilon\mathfrak{K}$, being called the carrier of k. The ring associated to a closed subset F is then the quotient ring of $\mathfrak{K}$ by the ideal of those elements of $\mathfrak{K}$ whose carriers do not meet F. This leads to the notions of "complexe" (not to be confused with a complex in the usual sense) and "couverture." A typical fundamental theorem is as follows: Let B be a faisceau of proper differential rings, $\mathfrak{X}$ a fine couverture of X. Then the cohomology ring of the reduced tensor product $\mathfrak{X}\bigcirc B$ is independent of the choice of $\mathfrak{X}$ and can be called the cohomology ring of the space X relative to the proper differential faisceau B. More general statements are made when the rings are filtered. Similarly, when $f: X\rightarrow Y$ is a continuous mapping, we take a proper differential faisceau B in X and fine couvertures $\mathfrak{X}$, $\mathfrak{Y}$ in X, Y, respectively. The cohomology ring $H(f^{-1}(\mathfrak{Y})\bigcirc\mathfrak{X}\bigcirc B)$ of the reduced tensor product is independent of the choice of $\mathfrak{X}$ and $\mathfrak{Y}$, and is called the ring of the mapping.

Chapter III proves the topological invariance of the cohomology rings when B is locally isomorphic to a given ring. Methods are given for the effective determination of the cohomology ring of a polyhedron and of that of a simplicial mapping. The author remarks that the ring of a mapping is not a homotopy invariant, but can be used to derive the Hopf invariant, the Steenrod functional products, and Gysin's theorem on sphere bundles.

According to the author, this general theory is the result of the study of homology properties of fiber spaces. In the opinion of the reviewer the main advantages of the theory consist in the systematic use of cochains instead of cocycles and cohomology classes and in the possibility of filtering the rings in a proper way. The work has contact with H. Cartan's homology theory and Koszul's work on the cohomology of homogeneous spaces. *S. Chern.*

Serre, Jean-Pierre. Cohomologie des extensions de groupes. C. R. Acad. Sci. Paris 231, 643–646 (1950).

Let A be a ring, G a group which operates on A, and g a normal subgroup of G. Lyndon [Duke Math. J. 15, 271–292 (1948); these Rev. 10, 10] has proved a theorem that gives relations between the cohomology rings $H(G, A)$, $H(g, A)$, and $H(G/g, A)$. By making use of the idea of the spectral homology ring and the filtered homology ring introduced by Leray [see the preceding review] and of H. Cartan's cohomology theory for spaces in which a group operates [same C. R. 226, 303–305 (1948); these Rev. 9, 368], the author is able to give a more precise form to

Lyndon's results. In the special case in which the subgroup g is a free group, the cup-product reduction theorem of Eilenberg and MacLane [Ann. of Math. (2) **48**, 51–78 (1947); these Rev. **8**, 367] is obtained as a corollary. Application is also made to the case in which there are given three commutative fields, $K\subset F\subset L$, where F and L are finite Galois extensions of K, and G and G/g are the Galois groups of L over K and F over K, respectively.

W. S. Massey (Providence, R. I.).

Whitehead, George W. The $(n+2)^{nd}$ homotopy group of the n-sphere. Ann. of Math. (2) **52**, 245–247 (1950).

As a consequence of results by Blakers and Massey [Proc. Nat. Acad. Sci. U. S. A. **35**, 322–328 (1949); these Rev. **11**, 47] the author proves that the suspension $\pi_4(S^2)\to\pi_5(S^3)$ is isomorphic. All $\pi_{n+2}(S^n)$ are cyclic of order 2, and $\pi_{11}(S^9)$ and $\pi_{12}(S^9)$ are nonzero. *H. Freudenthal* (Utrecht).

Seifert, Herbert. Closed integral curves in 3-space and isotopic two-dimensional deformations. Proc. Amer. Math. Soc. **1**, 287–302 (1950).

Let A_λ, $0\leqq\lambda\leqq1$, be an isotopy (isotopic deformation) of the Euclidean plane E. The closed curve $A_\lambda A_1 A_\lambda^{-1}(P)$, $0\leqq\lambda\leqq1$, is called the indicatrix of P. In general a point P does not lie on its indicatrix; the exceptions to this are precisely the trajectories f of the fixed points F of A_1. An unexceptional point P has, relative to its indicatrix, a certain order $\omega(P)$, called the rotation number of the deformation at P. Theorem 1: If U and V are unexceptional and if the fixed points F_ν of A_1 have no limit point, then $\omega(V)-\omega(U)=\sum_\nu\gamma_\nu S(f_\nu, q)$, where γ_ν is the fixed point index of F_ν under the mapping A_1, f_ν is the trajectory of F_ν, and $S(f_\nu, q)$ is the intersection number of f_ν and a curve q running from U to V. Theorem 2: Let A_λ be an ϵ-isotopy of E such that A_1 has at most isolated fixed points $F_1, F_2, \cdots$. If V is not on any of the trajectories $f_1, f_2, \cdots$ then $\omega(V)=\sum_\nu\gamma_\nu\theta_\nu$, where θ_ν is the order of V relative to f_ν. Theorem 3: If A_λ is an ϵ-isotopy of E and A_1 is without fixed points, then $\omega(P)=0$ for every P. These theorems may be generalized to the case where A_λ is only defined in a certain open region $G\subset E$. The results may be applied to the study of continuous vector fields in 3-dimensional fibre bundles; in particular, theorem 4: A continuous vector field on the 3-sphere which is approximately a field of Clifford parallels and which sends through every point exactly one integral curve has at least one closed integral curve. *R. H. Fox.*

Milnor, J. W. On the total curvature of knots. Ann. of Math. (2) **52**, 248–257 (1950).

According to a suggestion by R. H. Fox, the total curvature $\kappa(C)$ of any closed curve C in Euclidean n-space H^n is defined as sup $\kappa(P)$ for all polygons P inscribed in C, the total curvature $\kappa(P)$ of a polygon P being defined as the sum of the angles subtended between the prolongation of any of its sides and the next one. This definition is consistent in the case of C being itself a polygon, and in the case of C having a continuous curvature K it agrees with Fenchel's [Math. Ann. **101**, 238–252 (1929)] definition of $\kappa(C)=\int_C K ds$ extended to H^n by Borsuk [Ann. Soc. Polon. Math. **20** (1947), 251–265 (1948); these Rev. **10**, 60]. Fenchel's theorem, $\kappa(C)\geqq2\pi$, equality holding only for plane convex curves, is shown to be valid for general curves, and a proof of Borsuk's conjecture, $\kappa(C)>4\pi$, for the case of a knot in H^3, is given. For each closed curve C given by the variable vector $\mathfrak{x}(t)$ in H^n and each unit vector $\mathfrak{b}$, the author defines $\mu(C,\mathfrak{b})$ as the number of maxima of the projection $\mathfrak{b}\cdot\mathfrak{x}(t)$ and calls $\mu(C)=\min_{\mathfrak{b}}\{\mu(C,\mathfrak{b})\}$ the crookedness of C. This is a positive integer or ∞. It is related to $\kappa(C)$ by $2\pi\int_{S^{n-1}}\mu(C,\mathfrak{b})dS$, being equal to $\kappa(C)$ times the measure of S^{n-1}, when $\mathfrak{b}$ ranges over that unit sphere and the integral is taken in Lebesgue's sense; from this follows $\kappa(C)\geqq2\pi\mu(C)$. A curve type $\mathfrak{C}$ in H^n is an equivalence class of closed curves which can be carried into a member of the class by an isotopy of H^n. It is called simple, if the representative curves are simple. All circles belong to the same type, and this type is unknotted, all other types are knotted. A type is called tame if it contains a polygon, otherwise wild [definition by Fox and Artin, same Ann. (2) **49**, 979–990 (1948); these Rev. **10**, 317]. Put $\kappa(\mathfrak{C})=\inf\kappa(C)$ and $\mu(\mathfrak{C})=\min\mu(C)$, C ranging over $\mathfrak{C}$. If $\mu(\mathfrak{C})<\infty$, then $\mathfrak{C}$ is tame, and conversely. If C is a knot, then $\mu(C)\geqq2$ and $\kappa(C)>\kappa(\mathfrak{C})$, hence $\kappa(C)>4\pi$. For any simple curve type $\kappa(\mathfrak{C})=2\pi\mu(\mathfrak{C})$. Another consequence is this: Given a knot C in H^3 for which $\mu(C)<\infty$, there is a plane whose intersection with C consists of at least six components. *J. Nielsen* (Copenhagen).

Fox, R. H. On the total curvature of some tame knots. Ann. of Math. (2) **52**, 258–260 (1950).

This paper is closely related to an investigation of Milnor [see the preceding review]. The author takes into consideration the group $\mathfrak{G}$ of a tame knot in H^3, which is an invariant of the isotopy type $\mathfrak{C}$ of the knot, $\mathfrak{G}=\mathfrak{G}(\mathfrak{C})$, and for its rank ρ (minimum number of generators) the inequality $\rho(\mathfrak{C})\leqq\mu(\mathfrak{C})$ holds. Furthermore, the author derives from the structure of this group by algebraic means a new invariant $\lambda(\mathfrak{C})$, satisfying $\lambda(\mathfrak{C})\leqq\rho(\mathfrak{C})-1$ and thus also $1+\lambda(\mathfrak{C})\leqq\mu(\mathfrak{C})$. For any composite knot [J. W. Alexander, Trans. Amer. Math. Soc. **30**, 275–306 (1928)] the invariant λ of the composite knot is the sum of the λ's of the factors. For equal factors this yields $\mu(\mathfrak{C}^m)\geqq m+1$. The equality sign holds, for instance, if $\mathfrak{C}$ is the type of the clover leaf knot. This shows the existence of tame knot types with any prescribed integer value $2\leqq\mu<\infty$ of the crookedness. For the m-fold product of the type of the clover leaf knot with itself the total curvature takes the value $2\pi(m+1)$. The problem arises of the relationship between different invariants of an isotopy type of knots, for instance between the minimum number $\nu(\mathfrak{C})$ of crossings of the projection of knots of a certain type $\mathfrak{C}$ and the crookedness $\mu(\mathfrak{C})$. The question whether or not the inequality $\nu(\mathfrak{C})\geqq3(\mu(\mathfrak{C})-1)$ holds in general is suggested as a problem.

J. Nielsen (Copenhagen).

GEOMETRY

***Forder, H. G. Geometry.** Hutchinson's University Library, London; Longmans, Green and Co., Inc., New York, N. Y., 1950. 200 pp. $1.60.

The opening chapter, on Euclidean geometry, contains many theorems on circles, followed by a proof that any displacement in 3-space is the product of two half-turns, and consequently is a screw displacement. In the chapter introducing plane curves, a quartic is derived by modifying the "product" of two conics, with the result that the 28 bitangents are immediately visible. The two-cusped epicycloid is described as the bright curve seen "when the sun shines on a cup of tea." The chapter on projective geometry contains

a sketch of the proof that the whole theory of the projective plane can be derived from the simplest axioms of incidence along with Pappus's theorem. The fifth chapter is a concise exposition of non-Euclidean geometry. The chapter on logical structure stresses the abstract nature of the order relation (ABC) by comparing it with the human relation "A prefers C to B." The possibility of coordinatizing any descriptive geometry of three or more dimensions is epitomized in the statement that "we can create magnitudes from a mere muchness," and Archimedes' axiom in the statement that "you will always reach home, if you walk long enough." The chapter on solid geometry treats such topics as the connection between transversals of skew lines and Möbius tetrahedra. In the chapter on differential geometry one finds an account of parabolic points; also a proof of Liouville's theorem that every conformal transformation in three dimensions is a product of inversions in spheres, and some pictures of surfaces of constant curvature. The chapter on algebraic plane curves contains not only Plücker's equations but also the less familiar equation of Klein connecting the numbers of real singularities. The formal definition of genus is reinforced by the remark that this property of a curve is "the number of nodes and cusps it might have, but hasn't." There is also a discussion of higher singularities. In the chapter on representational geometry, a representation of oriented circles provides an original approach to Minkowskian space. There is also a version of the double-six theorem in terms of spheres. The final chapter, on general space, deals with Riemannian geometry, free mobility in the infinitesimal, groups and their fundamental regions, locally Euclidean spaces, distance geometries, and metric topology, all in fourteen pages! There is an extensive bibliography as well as an excellent index. *H. S. M. Coxeter* (Toronto, Ont.).

Schuster, Jan. Contribution à la géométrie du tétraèdre. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1948**, no. 16, 15 pp. (1949). (Czech. French summary)

If two equal vectors, opposite in sense, are taken on each edge of a tetrahedron, their resultants through each vertex lie on lines belonging to the same regulus of an hyperboloid of one sheet circumscribed about the tetrahedron. Other properties of those vectors are considered. The author obtains solutions of such classical problems as the conditions that the four lines through the vertices of a tetrahedron shall be hyperbolic, etc. There are no bibliographical references other than to an earlier paper of the author [same journal **1947**, no. 4 (1948); these Rev. **9**, 525]. *N. A. Court.*

Marmion, Alphonse. Sur la géométrie du tétraèdre. C. R. Acad. Sci. Paris **231**, 890–892 (1950).

Cavallaro, Vincenzo G. Funzioni continue e proposizioni geometriche concomitanti. Estensione del teorema di Pitagora. Boll. Un. Mat. Ital. (3) **5**, 174–177 (1950).

Guillotin, R. Sur le cercle pédal d'un point variable. Mathesis **59**, 90–94 (1950).

The paper deals with the locus of a point whose pedal triangle with regard to a given triangle passes through a given point. The locus is in general a sextic having triple points at the vertices of the triangle, which of course corresponds by a quadratic transformation to a cubic. The properties of the latter are developed, including the special cases where it is composite. *R. A. Johnson.*

Pernet, Roger. Anarotations relatives à un système de deux cercles. Ann. Univ. Lyon. Sect. A. (3) **12**, 27–32 (1949).

An anarotation is defined as the product of two spherical rotations in Euclidean space S_3. The paper treats the anarotations which transform one given circle into another, and those which interchange two given circles. *R. A. Johnson* (Brooklyn, N. Y.).

Dolaptschiew, Blagowest. Ein neues Verfahren zur Untersuchung der rechtwinkligen Projektion der Durchdringungskurve zweier Rotationsflächen 2-en Grades auf die Ebene ihrer Drehachsen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **37**, 319–362 (1941). (Bulgarian. German summary)

Nádeník, Zbyněk. Sur les courbes polaires de la cubique gauche. Časopis Pěst. Mat. Fys. **75**, D131–D139 (1950). (Czech. French summary)

If the points Y, Z are harmonic conjugates with respect to two points on the cubic curve c given by $x_i=t_1^{4-i}t_2^{i-1}$, $i=1, 2, 3, 4$, then Z is called the pole of Y with respect to c. If Y describes a curve k, Z describes the polar curve of k. The author studies the influence of various types of contact of k and c on the polar curve. He also finds necessary and sufficient conditions for a cubic curve k to be identical with its polar, if k and c have 4, 3, or 2 points in common. *F. A. Behrend* (Melbourne).

✶Jung, Heinrich W. E. Die Geraden einer Fläche zweiter Ordnung. Hallische Monographien no. 16, pp. 23–29. Max Niemeyer Verlag, Halle (Saale), 1950. 4.80 RM.

The conditions for a line with Plücker coordinates $(p_1, p_2, p_3, p_4, p_5, p_6)=(p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23})$ to be a generator of the quadric $\sum_{i,j=1}^{4} g_{ij}x_ix_j=0$ are the six linear equations (1) $\sigma p_i=\sum_{j=1}^{6} G_{ij}p_j$, where the 21 quantities G_{ij} are the quadratic minors of the symmetrical matrix g_{ij}. The relations between the matrices g_{ij}, G_{ij} (of rank 4, 6, respectively) are investigated and in particular it is shown that $|G-\sigma E|=(\sigma^2-|g|)^3$ (where E is the unit matrix). Equations (1) have thus two doubly infinite linear systems of solutions, corresponding to the characteristic roots $\sigma=\pm|g|^{\frac{1}{2}}$, and these (with the quadratic condition $p_1p_4+p_2p_5+p_3p_6=0$) give the two systems of generators of the quadric. *P. Du Val* (Athens, Ga.).

Gyarmathi, L. Konstruktive Lösung der Apollonius-Aufgabe im n-dimensionalen Raum durch Benützung einer Erweiterung der zyklographischen Abbildung auf mehrdimensionale Räume. Publ. Math. Debrecen **1**, 123–128 (1949).

The problem of Apollonius in n-dimensional space R_n is solved constructively by mapping the oriented hyperspheres in R_n with centre $(x_1, \cdots, x_n)$ and (positive or negative) radius r on the points $(x_1, \cdots, x_n, r)$ of an R_{n+1}. *F. A. Behrend* (Melbourne).

Hristov, Hr. Ya. On a relation between the volume of a simplex and the volumes of its boundaries. Doklady Akad. Nauk SSSR (N.S.) **73**, 25–28 (1950). (Russian)

Let m points $a_1, \cdots, a_m$ in E_n be given which lie in no E_{m-2}. Let U denote the volume of the simplex spanned by the a_i, and U_p, U_{pq}, U_{pqt} the volumes of the $(m-2)$-, $(m-3)$-, $(m-4)$-simplices obtained by leaving out a_p, a_p and a_q, a_p and a_q and a_t, respectively, where the volume of

a 0-simplex is put equal to 1, and the volume of an empty simplex equal to 0. Then the following relation holds:

$$\begin{aligned}(m-1)^4(m-3)^{-4}U^2U^2_{pqt}&+4(m-2)^2(m-3)^{-2}U_pU_qU_tU_{pqt}\\ &+4(m-1)^2(m-2)^{-2}UU_{pq}U_{qt}U_{tp}\\ &-(U_pU_{qt}+U_qU_{pt}+U_tU_{pq})^2\\ &+2(U^2_pU^2_{qt}+U^2_qU^2_{pt}+U^2_tU^2_{pq})\\ &-2(m-1)^2(m-3)^{-2}UU_{pqt}(U_pU_{qt}+U_qU_{pt}+U_tU_{pq})=0.\end{aligned}$$

H. Busemann (Los Angeles, Calif.).

Hajós, G. Translation of figures between lattice points. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 246–253 (1950).

Let L be a point-lattice in Euclidean n-space, and F a point set (e.g., a polytope) containing no point of L. Let F be displaced by translations in every possible way subject to the restriction that no point of L shall be covered. Let U be the class of points not covered by any of these positions of F. Then U includes L but does not necessarily coincide with L. For instance, let L consist of alternate points of an ordinary cubic lattice in 3-space, and let F be the open octahedron concentric with one of the cubes and of such a size that the centers of its faces are the vertices of the cube. Then the new positions of F are concentric with the other cubes, and U consists of the whole cubic lattice (having two points for every one in L). The author proves that such a phenomenon is impossible when F is a parallelepiped, but possible when F is a parallelotope in higher space ($n\geqq 4$).

H. S. M. Coxeter (Toronto, Ont.).

***Schmidt, Hermann. Die Inversion und ihre Anwendungen.** Verlag von R. Oldenbourg, München, 1950. 93 pp.

This is a useful account of many aspects of the subject of inversion in Euclidean geometry of two and three dimensions (parts A and B, respectively). In part A, chapter I is introductory and quite elementary; chapter II contains neat proofs of Ptolemy's theorem and Feuerbach's theorem; the next three chapters deal with various problems of construction, including constructions with compasses alone [C. Mohr, Euclides Danicus, Amsterdam, 1672, Høst, København, 1928]. These are followed by a short chapter on the linkages of Peaucellier and Hart, and a long one on Apollonius' problem: to draw a circle to touch three given circles. Chapter VIII, on higher plane curves, begins with the theorem that the inverse of a curve of order n is the pedal of a curve of class n. There is a thorough treatment of lemniscates, limaçons, and anallagmatic curves. Chapter IX deals briefly with the use of complex numbers.

In part B, chapter I deals with pencils of circles on a sphere; chapter II with the extension of Apollonius' problem to such circles; and chapter III with stereographic projection. Then comes a long and interesting chapter on the Dupin cyclide, followed by a short one on non-Euclidean geometry. Finally, part C contains three applications of inversion to the theory of electricity.

There are more than one hundred figures, all carefully drawn and some quite beautiful, especially the torus with its loxodromes on p. 82. *H. S. M. Coxeter.*

Terasaka, Hidetaka. A topological characterization of affine transformations in E^2. Osaka Math. J. **2**, 23–31 (1950).

Zur topologischen Charakterisierung der affinen Abbildungen einer Ebene auf sich führt Verf. 6 Axiome über lineare Abhängigkeit bzw. Unabhängigkeit von Punkten im n-dimensionalen Raum R_n ein. Nimmt man hierzu das Axiom, dass eine stetige Gruppe H von Abbildungen des R_n existiert, mit der Eigenschaft, dass genau eine Transformation der Gruppe existiert, die ein gegebenes $(n+1)$-Tupel linear unabhängiger Punkte in ein gegebenes $(n+1)$-Tupel linear unabhängiger Punkte überführt, so ist durch diese Axiome die Gruppe der affinen Abbildungen des R_2 auf sich charakterisiert. Verf. vermutet, dass die entsprechende Aussage auch für $n>2$ gilt. Zum Beweis für $n=2$ werden zunächst axiomatisch die topologischen Bilder der Geraden der Ebene eingeführt als die Gesamtheit von L-Linien, so dass durch irgend zwei verschiedene Punkte genau eine L-Linie geht und die Gesamtheit der L-Linien bei den Transformationen von H in sich übergeht. Bei geeigneter Definition des Begriffes Parallelismus bilden je drei Scharen paralleler L-Linien ein Sechseckgewebe. Das ermöglicht die Einführung einer Algebraisierung, die den L-Linien lineare Gleichungen zuordnet und zeigt, dass H mit der Gruppe der affinen Abbildungen der Ebene isomorph ist.

R. Moufang (Frankfurt am Main).

Bachmann, F. Geometrien mit euklidischer Metrik, in denen es zu jeder Geraden durch einen nicht auf ihr liegenden Punkt mehrere Nichtschneidende gibt. I. Math. Z. **51**, 752–768 (1949).

An example of a metric Euclidean geometry in which there is more than one parallel to each line through an exterior point has been given by Dehn with his semi-Euclidean geometry [Math. Ann. **53**, 404–439 (1900)]. A metric Euclidean geometry is defined as a geometry satisfying the axiom system of A. Schmidt [ibid. **118**, 609–625 (1943); these Rev. **6**, 13] and the axiom that every quadrilateral containing three right angles is a rectangle. (A geometry satisfying, in addition, the parallel axiom, is Euclidean.) Consider a Euclidean geometry G and a metric Euclidean geometry M. If every point of M is a point of G and if every line of G which passes through a point of M is a line of M, then M is said to belong to G. It is shown that a subset S of points and lines of a Euclidean geometry G constitutes a metric Euclidean geometry which belongs to G if and only if (1) there are at least two points in S, (2) corresponding to the points P_1, P_2, P_3 in S, there exists a point P_4 in S such that the reflection equation $P_1P_2P_3=P_4$ holds, (3) if A and B are points of S and k is an arbitrary line of S through A, then the point of intersection of k and the line through B orthogonal to k is a point of S. In a Euclidean geometry G in which the right angle is bisectable, it is shown that a subset S of points constitutes a metric Euclidean geometry which belongs to G if and only if (1) there are at least two points in S, (2) if A and B are points of S, then every point of the circumference of the circle on AB as diameter is a point of S.

If K is the coordinate field of a Euclidean geometry G, then $C(K)$ is used to denote the ring produced by the elements $(1+c^2)^{-1}$ with c in K. It is shown that $C(K)$ is the coordinate set of the smallest metric Euclidean geometry belonging to G and containing the points $(0, 0)$ and $(1, 0)$. Consider the following fundamental constructions: (1) to draw the line determined by two given points; (2) to draw the perpendicular to a given line through a given point, (3) to construct the point of intersection of two given orthogonal lines. It is shown that, if we proceed from the points $(0, 0)$, $(1, 0)$, and $(0, 1)$ in ordinary real Euclidean geometry, we can construct, using (1), (2), and (3), only those points of which the coordinates belong to $C(P)$, where P is the field of rational numbers. *L. Dulmage.*

Bachmann, F. Geometrien mit euklidischer Metrik, in denen es zu jeder Geraden durch einen nicht auf ihr liegenden Punkt mehrere Nichtschneidende gibt. II. Math. Z. **51**, 769–779 (1949).

If G is a Euclidean geometry over a field K, it is shown that the operation $(1+u^2)^{\frac{1}{2}}$ is feasible for all u in K if and only if every angle in G is bisectable. A field in which -1 is not a square and in which the operation $(1+u^2)^{\frac{1}{2}}$ is feasible for all u is called a Hilbertian field. The symbol Ω is used to denote the smallest Hilbertian field. It is shown that the ring $C(\Omega)$ is identical with Ω. This implies that there exists no proper metric Euclidean subgeometry belonging to the Euclidean geometry over the field Ω. The symbol $\Omega(t)$ is used to denote the Hilbertian field consisting of all those algebraic functions which result from the parameter t through the rational operations and the operation $(1+u^2)^{\frac{1}{2}}$. It is shown that the geometry which has the ring $C(\Omega(t))$ as coordinate set is a proper subgeometry of Dehn's semi-Euclidean geometry. This geometry is a metric Euclidean geometry which belongs to the Euclidean geometry over the field $\Omega(t)$.

To the fundamental constructions (1), (2), and (3) of the paper reviewed above a fourth fundamental construction is added: (4) given a point A on a line g, to construct a point B on g so that the segment AB is congruent to the unit segment. It is shown that, if we proceed from the points $(0, 0)$ and $(1, 0)$ in a Euclidean geometry with angle bisectability, we can construct, using (1), (2), (3), and (4) those and only those points of which the coordinates are elements of Ω. A criterion is found also for those points which can be constructed using (1), (2), (3), and (4) proceeding in Euclidean geometry with angle bisectability from the points $(0, 0)$, $(1, 0)$, and $(0, t)$, where t is a parameter.

L. Dulmage (Kingston, Ont.).

Busemann, Herbert. Non-Euclidean geometry. Math. Mag. **24**, 19–34 (1950).

This is a very readable introduction to non-Euclidean geometry, based on the mobility principle of Helmholtz. The essential step is a neat proof that the length t of the arc of an equidistant curve at distance b from a straight line is related to the corresponding distance s along the line by an equation of the form $t=s\phi(b)$, where $\phi(b)$ satisfies the functional equation $\phi(x+y)+\phi(x-y)=2\phi(x)\phi(y)$. The only serious misprint is "Bolay" for "Bolyai" at the beginning.

H. S. M. Coxeter (Toronto, Ont.).

Kagan, V. F. The development of the interpretation of non-Euclidean geometry. Trudy Sem. Vektor. Tenzor. Analizu 7, 187–204 (1949). (Russian)

A general type of transformations T_n of the hyperbolic plane in the interior of a Euclidean circle is discussed. The transformation T_n transforms straight H-lines into algebraic curves of the nth order. It appears that for n even and n odd the characteristic properties are different. For $n=1, 2$ the well-known models of Klein and Poincaré are obtained. Finally, the metric properties and the appropriate group of movements are studied. *H. A. Lauwerier* (Amsterdam).

Petkančin", B. Geometries and relations between them. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **38**, 193–216 (1942). (Bulgarian)

Löbell, Frank. "Landkarten" der nichteuklidischen Ebene. Jber. Deutsch. Math. Verein. **54**, 4–23 (1950).

Der Verfasser leitet die bekannten Abbildungen der Lobatschefskischen Ebenen her aus einem einheitlichen Gesichtspunkt. In derselben Weise wie man Kartenprojektionen der Erde zu konstruieren pflegt, entwirft er Bilder der hyperbolischen Ebene in der Euklidischen Ebene: (1) strahlige Entwürfe, flächentreu, winkeltreu, usw.; das Poincarésche und das Kleinsche Modell werden hier wiedergefunden; (2) zylindrische Entwürfe, flächentreu, winkeltreu, usw.; (3) andere Entwurfsarten, welche in der Kartenentwurfslehre kein Analogon besitzen. Zum Schluss werden noch einige Abbildungen auf eine nicht-Euklidische Ebene besprochen. *H. A. Lauwerier* (Amsterdam).

Garnier, René. Sur les axoïdes et la viration dans les espaces cayleyens. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 305–323 (1950).

The general infinitesimal motion of a rigid solid is a screw displacement having a definite line for its axis. Thus a continuous motion has an instantaneous axis at every stage. The locus of this axis in space is a ruled surface: the polhode. Its locus in the body is another ruled surface: the herpolhode. The herpolhode rolls on the polhode while sliding along the generator of contact. The author extends these classical results to non-Euclidean spaces. In the elliptic case there is the slight complication that the infinitesimal motion has two axes (absolute polars). In the hyperbolic case there is the more serious complication that the infinitesimal motion may be a parabolic rotation (or parallel displacement) having no accessible axis; thus the polhode and herpolhode may be real but inaccessible. *H. S. M. Coxeter* (Toronto, Ont.).

Szász, Paul. Neue Herleitung der hyperbolischen Trigonometrie in der Ebene. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 44–52 (1950).

A derivation of hyperbolic geometry, new in some details, is given. By using elementary methods the author derives for the geometrically defined goniometric functions $S(x)$ and $C(x)$ the functional relations $S(2x)=2S(x)C(x)$ and $S(x)^2+C(x)^2=1$. From this and from the fact that $\lim_{x\to 0} S(x)/x$ exists, it follows that these functions are identical with the sine and cosine functions.

H. A. Lauwerier (Amsterdam).

Algebraic Geometry

Arvesen, Ole Peder. Sur l'étude de certaines courbes algébriques comme courbes polaires par rapport à n points. Norske Vid. Selsk. Forh., Trondhjem **20**, no. 9, 33–36 (1948).

This paper contains the following results. The first polar $F=0$ of a line d with respect to a set of n points P_i has the $\frac{1}{2}n(n-1)$ lines joining the P_i as tangents. If $r+1$ of the P_i lie on d, d is an r-fold tangent to F. The problem of determining the r points of contact of d and F is equivalent to that of finding the reducible polars of the $r+1$ points P_i that lie on d. *T. R. Hollcroft* (Aurora, N. Y.).

Gaeta, Federico. Sulle famiglie di curve sghembe algebriche. Boll. Un. Mat. Ital. (3) **5**, 149–156 (1950).

Expository lecture.

Ichida, Asajiro. On the foci of algebraic curves. Proc. Japan Acad. **25**, no. 8, 1–6 (1949).

Generalizing the dual of the well-known theorem, the intersections of the tangents to an algebraic plane curve of

class m from the circular points at infinity, G and H, are the m^2 foci of the curve, for any two lines, g and h, intersecting curves of a pencil of order n in points P, Q, the author obtains the envelope of the lines PQ and also the locus of the intersections of the lines P_iQ_i and P_jQ_j. He writes the duals of this theorem where the points G and H are (1) any two points, (2) any two points at infinity, (3) the circular points at infinity. In the third case, the following theorem results: The locus of the foci of the curves of an algebraic pencil of class m is a circular algebraic curve of order $2m-1$ with each of the circular points at infinity $(m-1)$-fold, and the envelope of the lines joining the foci in pairs is a parabolic algebraic curve of class m^2-1 for which the line at infinity is an $(m-1)^2$-fold tangent. The theorems are generalized for S_3 and S_r. *T. R. Hollcroft* (Aurora, N. Y.).

Galbură, Gh. On algebraic irrational involutions. Acad. Repub. Pop. Romîne. Bul. Şti. A. 1, 551–554 (1949). (Romanian. Russian and French summaries)

Soit C une courbe algébrique, de genre p; I et J deux involutions irrationelles de genre π et ρ, sur C. Dans cette note on établit que le nombre s des couples des points, sur C, appartenant simultanément à un groupe de l'involution I et à un groupe de l'involution J, est fourni par $s=mn+m(\pi-1)+n(\rho-1)-(p-1)-mn\pi\rho/p$.
Author's summary.

Coşniţă, Cezar. Sur les sections circulaires des cyclides. Acad. Républ. Pop. Roum. Bull. Sect. Sci. 30, 459–461 (1948).

The locus of the centre of a circle variable in any of the ten families on a cyclide (by which is meant either a bispherical quartic or a spherical cubic surface) is a rational quartic [cubic] curve, and that of the axis (perpendicular to the plane at the centre) of the circle is a hyperboloid of one sheet [hyperbolic paraboloid], the curve being the locus of orthogonal projections of the vertex of the cone enveloped by the planes of the circles, onto the generators of the quadric. (Expressions in [ ] refer to the case in which the cyclide is cubic.) *P. Du Val* (Athens, Ga.).

Godeaux, Lucien. Sur une courbe algébrique gauche du sixième ordre. Mathesis 59, 156–161 (1950).

The three-binodal cubic surface $X_1X_2X_3=X_0^3$ is osculated along the sextic curve C (of genus 3):

$$\begin{Vmatrix} a_1X_1 & X_0 & 0 & -X_3 \\ a_2X_2 & -X_1 & X_0 & 0 \\ a_3X_3 & 0 & -X_2 & X_0 \end{Vmatrix}=0$$

by the sextic surface

$$\begin{aligned} &a_1^3X_1^5X_2+a_2^3X_2^5X_3+a_3^3X_3^5X_1 \\ &\quad+3X_0(a_1^2a_2X_1^3X_2^2+a_2^2a_3X_2^3X_3^2+a_3^2a_1X_3^3X_1^2) \\ &\quad+3X_0^2(a_1^2a_3X_1^3X_3+a_2^2a_1X_2^3X_1+a_3^2a_2X_3^3X_2) \\ &\qquad\qquad+6a_1a_2a_3X_0^6=0, \end{aligned}$$

which has 21 binodes on the curve C. The two mappings

$$\begin{aligned} X_0:X_1:X_2:X_3&=Y_1Y_2Y_3:Y_2^2Y_3:Y_3^2Y_1:Y_1^2Y_2 \\ &=y_1y_2y_3:y_1^2y_3:y_2^2y_1:y_3^2y_2 \end{aligned}$$

of the cubic surface on a plane transform the curve into

$$\begin{aligned} a_1Y_2^3Y_3^2+a_2Y_3^3Y_1^2+a_3Y_1^3Y_2^2&=0, \\ a_1y_1^3y_3+a_2y_2^3y_1+a_3y_3^3y_2&=0, \end{aligned}$$

respectively, and the $1-3$ transformation $X_0:X_1:X_2:X_3=x_1x_2x_3:x_1^3:x_2^3:x_3^3$ transforms it into the cubic involution generated by the homography $x_1':x_2':x_3'=x_1:\eta x_2:\eta^2x_3$ (where η is a primitive cube root of unity) on the curve $a_1x_1^5x_2+a_2x_2^5x_3+a_3x_3^5x_1=0$. The curve C is transformed into itself by the homography of period 7, $X_0':X_1':X_2':X_3'=\zeta^6X_0:X_1:\zeta X_2:\zeta^3X_3$ (where ζ is a primitive seventh root of unity). In the special case $a_1=a_2=a_3$, the equivalent quartic becomes Klein's quartic $y_1^3y_3+y_2^3y_1+y_3^3y_2=0$, and the homography of period 7 together with $X_0':X_1':X_2':X_3'=X_0:X_2:X_3:X_1$ generate the simple group of order 168.
P. Du Val (Athens, Ga.).

Godeaux, Lucien. Sur la structure des points unis d'une involution appartenant à la surface des couples de points d'une courbe algébrique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 383–387 (1950).

Godeaux, Lucien. Sur le système canonique de certaines surfaces de genre linéaire un. Bull. Soc. Roy. Sci. Liège 19, 204–212 (1950).

The equation is given of a surface F of order $3\nu+3$ which osculates the three-binodal cubic Φ: $X_1X_2X_3=X_0^3$ along the curve (of order $3\nu+3$ and genus $\frac{3}{2}\nu(\nu+1)$) C:

$$\begin{Vmatrix} a_1X_1^\nu & X_0 & 0 & -X_3 \\ a_2X_2^\nu & -X_1 & X_0 & 0 \\ a_3X_3^\nu & 0 & -X_2 & X_0 \end{Vmatrix}=0.$$

F, ϕ, and C are alike invariant under the periodic homography (of period $q=3\nu^2+3\nu+1$)

$$X_0':X_1':X_2':X_3'=\zeta^{-\nu}X_0:X_1:\zeta X^2:\zeta^{3\nu^2}X_3,$$

where ζ is a primitive qth root of unity, which determines an involution I^q on F. The surface F is regular, and has genera $p_g=p_a=\frac{1}{2}\nu(3\nu+1)(3\nu+2)$, $p^{(1)}=3(\nu+1)(3\nu-1)^2+1$. An image F' of the involution I^q is regular and has genera $p_g=p_a=\nu$, $p^{(1)}=1$. Its canonical system is compounded with a pencil of elliptic curves, images of the intersections of F with the pencil of cubics $X_1X_2X_3+\lambda X_0^3=0$, and has also the elliptic curve, image of the intersection of F with $X_0=0$, counted twice, as fixed part. *P. Du Val* (Athens, Ga.).

Godeaux, Lucien. Remarques sur les surfaces inscrites dans une surface cubique. Bull. Soc. Roy. Sci. Liège 19, 150–157 (1950).

If a surface F, of order $2n$, touches a nonruled cubic surface Φ along a curve Γ of order $3n$, Φ must be four-nodal, say $X_1X_2X_3+X_0X_2X_3+X_0X_3X_1+X_0X_1X_2=0$; for every curve of contact Γ the tangent surface F can be taken in the form $X_2X_3\varphi_1^2+X_3X_1\varphi_2^2+X_1X_2\varphi_3^2+X_0X_1(\varphi_2-\varphi_3)^2+X_0X_2(\varphi_3-\varphi_1)^2+X_0X_3(\varphi_1-\varphi_2)^2=0$, where φ_1, φ_2, φ_3 are arbitrary forms of order $n-1$ in (X_0, X_2, X_3), (X_0, X_3, X_1), (X_0, X_1, X_2), respectively. The surface F has $6n^2-9n+4$ conical double points on Γ, and $(n-1)^3$ at the points $\varphi_1=\varphi_2=\varphi_3=0$. The curves Γ form a system of dimension $\frac{3}{2}n(n+1)$ on the surface.

If a surface F', of order $3n$, osculates a nonruled cubic surface Φ' along a curve Γ' of order $3n$, Φ' must be three-binodal, say $X_0^3=X_1X_2X_3$; for every curve of contact Γ the osculating surface F' can be taken either in the form

$$\begin{aligned} &X_1^2X_2\alpha_1^3+X_2^2X_3\alpha_2^3+X_3^2X_1\alpha_3^3 \\ &\quad+3X_0(X_1X_2\alpha_1^2\alpha_2+X_2X_3\alpha_2^2\alpha_3+X_3X_1\alpha_3^2\alpha_1) \\ &\quad+3X_0^2(X_1\alpha_1^2\alpha_2+X_2\alpha_2^2\alpha_3+X_3\alpha_3^2\alpha_1)+6X_0^3\alpha_1\alpha_2\alpha_3=0 \end{aligned}$$

(where α_1, α_2, α_3 are forms of order $n-1$), or in a similar form with odd permutation of the suffixes (1, 2, 3) throughout. The curves Γ' form accordingly two distinct systems on Φ', each of dimension $\frac{3}{2}n(n+1)-1$; F' has $3(3n^2-3n+1)$ binodes on Γ', and $(n-1)^3$ conical triple points at $\alpha_1=\alpha_2=\alpha_3=0$.

The method in both cases involves expressing Φ or Φ' as the image of a plane involution (quadratic and cubic, respectively) with only isolated coincidences, and Γ or Γ' as the image of a plane curve belonging to the involution. [The primes on F', Φ', Γ' have been added by the reviewer to avoid confusion between the two sections of the paper, which are quite distinct and independent.] *P. Du Val.*

Godeaux, Lucien. Sur une surface multiple possédant des points de diramation quintuples. Bull. Soc. Roy. Sci. Liège 19, 83–96 (1950).

An algebraic surface F is considered, having a cyclic birational self transformation T of order 31, whose fixed points A are finite in number and all of them simple on F, and such that, in the tangent plane at each, T determines a homography of the form $x_0':x_1':x_2'=x_0:\epsilon x_1:\epsilon^{18}x_2$, where ϵ is a primitive 31st root of unity, and A is $(1, 0, 0)$. If $|C_0|$ is a linear system compounded with the involution generated by T, its projective model Φ is a 31-ple surface branching only at the images of the fixed points A, each of which is quintuple on Φ, the tangent cone consisting of three planes and a quadric cone (the latter meets two of the three planes each in a line, and one of these meets the third plane in a line). The partial neighborhoods and singular points in the neighborhood of this singularity are analyzed in great detail. The canonical system $|K'|$ on Φ is the image of a system $|K|$ on F, contained in the canonical system, having a sextuple base point at each point A, and further base points of multiplicities 4, 2, 2 consecutive to A along a cuspidal branch of the second kind, and 2, 2, 1, 1 consecutive to A along a cuspidal branch of the third kind (the tangents to these two branches being $x_1=0$, $x_2=0$). Hence if the number of points A is χ, the linear genera $p^{(1)}$, $\pi^{(1)}$, the Zeuthen-Segre invariants I, I', and the arithmetic genera p_a, π_a of F, Φ, respectively, satisfy the relations

$$p^{(1)}-1=31(\pi^{(1)}-1)+70\chi,$$
$$I=31I'-209\chi+120,$$
$$12(p_a+1)=12\cdot 31(\pi_a+1)-139\chi.$$

Finally the equations are given of an actual example (in 19 dimensions) of a surface F for which T is a homography, and its fixed points are all of the kind considered.

P. Du Val (Athens, Ga.).

Galafassi, Vittorio Emanuele. I risultanti nella topologia degli spazi proiettivi. Ricerca Sci. 20, 307–309 (1950).

The author considers the resultant R of $r+1$ algebraic forms F_i of respective orders n_i in S_r. Assuming that the F_i have no points in common so that $R\neq 0$, he obtains the following topological interpretation of the sign of R: The resultant R, suitably normalized, of $r+1$ real forms (generically located in a real S_r), at least two of which are of even order, is positive or negative according as the forms are nonseparable or separable, respectively. *T. R. Hollcroft.*

Fadini, Angelo. Gli S_r n-duali e le varietà di Segre degli S_p biduali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 557–562 (1950).

The "n-dual algebra" here considered is the hypercomplex system defined over the complex field by the units $u_1, u_2, \cdots, u_n$ which have the multiplication table $u_iu_1=u_1u_i=u_i$, $i=1, 2, \cdots, n$; $u_iu_j=u_ju_i=0$, $i, j=2, 3, \cdots, n$. A matrix representation of this algebra reveals that the numerical projective space of dimension r defined over this algebra (the n-dual S_r) can be interpreted as a subvariety of the variety of Segre associated with $(n-1)$ bidual projective spaces of dimension r. Representations of the n-dual S_r as complexes of subspaces of complex projective spaces are also given. *H. T. Muhly* (Iowa City, Iowa).

Andreotti, Aldo. Sopra alcune superficie algebriche uniformizzabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 569–570 (1950).

This note contains historical remarks on the problem of uniformization in the large for algebraic surfaces.

H. T. Muhly (Iowa City, Iowa).

***Garnier, René. Intégration uniforme de certains systèmes du quatrième ordre, à deux variables indépendantes, attachés à une surface algébrique.** Colloque de géométrie algébrique, Liège, 1949, pp. 105–121. Georges Thone, Liège; Masson et Cie., Paris, 1950. 200 Belgian francs; 1400 French francs.

The author considers the problem, first considered by Picard, of finding an algebraic surface $F(x_1, x_2, x_3)=0$ and rational functions $R_{ijk}(x_1, x_2, x_3)$ on the surface such that the system of equations (assumed to satisfy the conditions of integrability) $\partial x_i/\partial u_j=w_{ij}$, $\partial w_{ij}/\partial x_k=w_{ij}R_{ijk}(x_1, x_2, x_3)$ $(i, j, k=1, 2)$ has its general integral uniform. In the case where the differentials du_j have no essential singularities on the surface the author shows that the geometric genus $p_g qF$ cannot exceed unity. He obtains solutions of the problem for a number of surfaces all of which are elliptic or hyperelliptic, with the exception of elliptic ruled surfaces. The enumeration of possible surfaces F is complete if one assumes the truth of Enriques' statement that every surface with $p_g=1$, $p_a=0$ contains an elliptic pencil of curves of genus one, but the author implies that this result has not yet been established with complete rigour. If du and dv possess essential singularities on F the same surfaces arise if $p_g=1$, but difficulties arise when $p_g=0$. *J. A. Todd.*

Muhly, H. T., and Zariski, O. Hilbert's characteristic function and the arithmetic genus of an algebraic variety. Trans. Amer. Math. Soc. 69, 78–88 (1950).

The Hilbert function $\chi(m)$ of an algebraic variety V_r in S_n is the maximum number of linear members of the linear system on V_r cut by the hypersurfaces of S_n of order m. It is known that if m is sufficiently large $\chi(m)=\sum_{i=0}^{r}a_i\binom{m}{i}$, where the a_i are integers independent of m. The virtual arithmetic genus $p_a(V_r)$ of the locally normal variety V_r is defined to be $(-1)^r(a_0-1)$.

It is shown that if U and V are birationally equivalent locally normal varieties such that the transformations $T: U\rightarrow V$ and $T^{-1}: V\rightarrow U$ are "proper" transformations, $p_a(U)=p_a(V)$. If the transforms of almost all sections of U by linear spaces are locally normal, then T is a proper transformation. A regular transformation between birationally equivalent normal varieties is proper; hence the result implies that the virtual arithmetic genus of a normal variety is a relative birational invariant. In the case of varieties of dimension $r\leqq 3$, it is shown that a quadratic transformation, or a nonsingular monoidal transformation of a normal variety is proper. From this it is proved that all nonsingular models of a function field Σ of dimension r $(r\leqq 3)$ have the same virtual arithmetic genus; this common value is called the arithmetic genus of Σ, denoted by $p_a(\Sigma)$.

If $r=2$, it is shown that if U and V are normal models of the function field Σ and if the correspondence has no fundamental points on V, then $p_a(U)\geqq p_a(V)$. It then follows that, for any normal model U of Σ, $p_a(U)=p_a(\Sigma)+s$ $(s\geqq 0)$. It is then shown that if the general hyperplane

section C of U is of genus π, and if ν is the order of U, then $\chi(m)=\nu\binom{m}{2}+(\nu-\pi+1)m+p_a(\Sigma)+s$, provided that m is sufficiently large. This is the Riemann-Roch theorem for the system $|mC|$. The ground field is assumed to be algebraically closed, but is otherwise arbitrary. *W. V. D. Hodge.*

Seidenberg, A. The hyperplane sections of normal varieties. Trans. Amer. Math. Soc. 69, 357–386 (1950).

Le but de ce mémoire est de montrer que, étant donnée une variété normale (c.à.d. partout localement normale) V^r ($r\geqq 2$), presque toutes les sections hyperplanes de V sont normales ("presque toutes" signifie que les hyperplans H tels que $V\cap H$ ne soit pas normale font partie d'un sous-ensemble algébrique propre de l'espace des hyperplans). La démonstration se divise en trois parties (a) montrer que, presque toujours (p.t.) $H\cap V$ est irréductible; (b) montrer que, p.t., $H\cap V$ n'a pas de singularités de dimension $r-2$; (c) montrer que, p.t., dans l'anneau de coordonnées de $H\cap V$, tout idéal principal est équidimensionnel (c'est à dire n'a pas de composantes immergées); il est en effet montré que "(b) et (c)" équivaut à la normalité de $H\cap V$. Pour (b), on montre que, p.t., les singularités d'une section hyperplane d'une variété quelconque W sont induites par celles de W, ceci aussi bien au sens géométrique qu'à celui de la théorie des idéaux; la définition adoptée pour les points simples est celle de Zariski [mêmes Trans. 62, 1–52 (1947); ces Rev. 9, 99]: l'anneau local d'un point simple est régulier. On introduit alors la "forme associée" de W^s, c.à.d. l'équation $E=0$ de la projection générique de W sur un sous-espace de dimension $s+L$, et on montre que celle de $V\cap H$ est p.t. spécialisation de celle de $V\cap H_g$ (H_g: hyperplan générique). Pour (a) est introduite la notion suivante: un idéal premier P d'un anneau de polynomes $k[x_1, \cdots, x_n]$ est dit quasi absolument irréductible (q.a.i.) si, pour toute extension de k, l'idéal engendré par P est primaire; la variété correspondante est aussi dite q.a.i.; il revient au même de dire que, dans le corps des fractions de $k[x]/P$ tout élément algébrique sur k est p-radiciel; si V^r est q.a.i. et si $r\geqq 2$, la section hyperplane générique de V est q.a.i.; on en déduit, en examinant les formes associées, que, p.t., $H\cap V$ est q.a.i., c'est à dire que l'idéal engendré par celui de V et l'équation de H est p.t. premier et q.a.i. Pour (c) on remarque qu'une certaine dérivée partielle de la forme associée de $V\cap H$ a son image canonique d dans le conducteur de la fermeture intégrale de l'anneau de coordonnées R de $V\cap H$; si R n'était pas intégralement clos, d ne serait pas équidimensionnel; on montre donc que d est p.t. équidimensionnel, ce qui se déduit du fait que $V\cap H_g$ est normale. La démonstration de (c) suppose que le corps des fractions rationnelles sur V est séparable sur le corps de base k (remarques qu'ici les variétés ne sont pas toujours définies, à la Weil, par des extensions régulières); on se débarasse de cette hypothèse par des considérations fort techniques sur les conducteurs de fermetures intégrales. Pour des raisons de commodité le corps de base a été supposé infini; il est indiqué comment on pourrait se débarasser de cette hypothèse. Dans tout ceci est fait usage constant d'une généralisation d'un théorème de Krull [Arch. Math. 1, 56–64, 129–137 (1948); ces Rev. 10, 178; 11, 310]: si A est un idéal équidimensionnel de dimension q d'un anneau de polynomes $k(u_1, \cdots, u_s)[X_1, \cdots, X_r]$ sur un corps de fractions rationnelles $k(u)$, alors pour p.t. spécialisation $u_i\rightarrow a_i\varepsilon k$ des paramètres l'idéal de $k[X]$ obtenu à partir de A est équidimensionnel de dimension q; ce résultat est démontré en appendice par réduction à la dimension 1. Ce mémoire contient de nombreuses fautes d'impression; celles-ci ne géneront pas le lecteur s'il y apporte un peu d'attention et de bonne volonté. *P. Samuel* (Clermont-Ferrand).

***de Rham, Georges, and Kodaira, Kunihiko. Harmonic Integrals.** Institute for Advanced Study, Princeton, N. J., 1950. iii+114 pp.

These lectures, delivered during the spring term, 1950, at the Institute for Advanced Study, consist of five chapters, the first four, by the first author, giving an account of the most modern developments in the general theory of harmonic integrals, and the fifth, by Kodaira, applying the theory to certain problems concerning Kähler manifolds.

The first part demonstrates the remarkable advances that have been made in the theory of harmonic integrals since the idea was first developed. Many of the details and restrictions which originally encumbered the theory have been eliminated, and the theory now takes on a streamlined character and at the same time acquires much greater generality and width of application. Apart from the great simplification of presentation, the theory is generalised in two ways: First, it applies to nonorientable manifolds, and, second, the notion of an exterior differential form is replaced by the more general concept of a current, which bears to a form a relation similar to that which a distribution, in the sense of Schwartz, bears to a function. The general concept of current, and in particular of a harmonic current, is defined in any Riemannian space, but the main existence theorem, and the related theory in the large, is established only in the case of compact spaces.

The first chapter is devoted to the theory of exterior forms, and the most important advantage of the treatment given here over the more usual treatment is the introduction of a second kind of form, called of odd kind in contrast with the more usual type of form which is of even kind. If $\sum a_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}$ is a form expressed in terms of the coordinate system $(x^1, \cdots, x^n)$ and $\sum b_{i_1\cdots i_p}dy^{i_1}\cdots dy^{i_p}$ is the same form expressed in terms of the coordinate system $(y^1, \cdots, y^n)$, then

$$b_{i_1\cdots i_p}=\sum_j a_{j_1\cdots j_p}\frac{\partial(x_{j_1}, \cdots, x_{j_p})}{\partial(y_{i_1}, \cdots, y_{i_p})}$$

or

$$b_{i_1\cdots i_p}=\frac{J}{|J|}\sum_j a_{j_1\cdots j_p}\frac{\partial(x_{j_1}, \cdots, x_{j_p})}{\partial(y_{i_1}, \cdots, y_{i_p})},$$

where J is the Jacobian of the transformation of coordinates, according as the form is of even or of odd kind. Correspondingly, two kinds of cells, and ultimately of chains, are defined, the usual type of cell or chain being of odd kind. Forms of even (odd) kind are integrated over chains of odd (even) kind. Then the classical results of the theory of forms, including such theorems as Stokes' theorem, usually given only for forms of even kind, are developed for forms of both kinds.

Chapter II introduces the notion of a current (of even or odd kind), which is a linear functional of the forms C^∞ (of odd or even kind), and which can be written symbolically as a form whose coefficients are distributions. These enable us to include in a single theory various diverse objects, including, for example, forms and chains (a form of multiplicity p being a current of degree $n-p$, and a chain of dimension p being a current of degree p); and various operations usually defined only for forms are defined for currents. Chapter III considers Riemannian spaces, and the definition of a harmonic form is extended to give a definition

of a harmonic current. The usual theory of harmonic forms in a compact orientable Riemannian space is then generalised to give similar results for harmonic currents in any compact Riemannian space, the results for nonorientable spaces yielding important new properties of such spaces. This chapter ends by using the theory to give a very general theory of the Kronecker index of chains and cycles. The fourth chapter introduces the notion of homotopic currents, including as a special case the homotopy theory of chains. The idea is used to show that any current can be approximated arbitrarily closely by a form C^∞.

In the second part of these lectures, the general theory is applied to problems concerning the existence of meromorphic functions on a Kähler manifold. Conditions are obtained for the existence of many-valued meromorphic functions with prescribed singularities, of one-valued meromorphic functions with prescribed singularities (generalised Abel's theorem), and of generalised theta functions.

W. V. D. Hodge (Cambridge, England).

Differential Geometry

Popa, Ilie. Periodic solutions of a system of differential equations and conditions for the closure of a plane curve. Acad. Repub. Pop. Romåne. Bul. Şti. A. **1**, 539–541 (1949). (Romanian. Russian and French summaries)

L'auteur s'occupe des conditions requises pour qu'une courbe plane donnée, d'équation intrinsèque $1/\rho=f(s)$ soit fermée. Il arrive au système

$$d\alpha/ds-\beta f(s)-1=0,\quad d\beta/ds+\alpha f(s)=0$$

et montre que ces conditions sont données par

$$\int_\Gamma dx=\int_0^{2\pi}\rho\cos\varphi d\varphi=0,\quad \int_\Gamma dy=\int_0^{2\pi}\rho\sin\varphi d\varphi=0.$$

Author's summary.

Bacon, Ralph Hoyt. The pursuit course. J. Appl. Phys. **21**, 1065–1066 (1950).

Buch, Kai Rander. An elementary pursuit problem. Mat. Tidsskr. B. **1950**, 128–130 (1950). (Danish)

Tzénoff, Iv. Sur les points simples et singuliers des courbes gauches. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **37**, 131–148 (1941). (Bulgarian. French summary)

Yaglom, I. M. The tangential metric in a two-parameter family of curves in the plane. Trudy Sem. Vektor. Tenzor. Analizu **7**, 341–361 (1949). (Russian)

Let $F(u, v, \xi, \eta)$ represent a two-parameter family of curves in the plane in line coordinates u, v (ξ and η are the parameters). Let $u=U(\xi, \eta, d\eta/d\xi)$, $v=V(\xi, \eta, d\eta/d\xi)$ be the solutions of the conditions $F(u, v, \xi, \eta)=0$ and $F(u, v, \xi+d\xi, \eta+d\eta)=0$ for two infinitely near curves to have u, v as common tangent. The tangential distance of two curves of the family is defined by

$$d\sigma^2=(u^2+v^2)[(F_uF_{v\xi}-F_vF_{u\xi})d\xi \\ -(F_vF_{u\eta}-F_uF_{v\eta})d\eta]^2(uF_u+vF_v)^{-4},$$

where u and v are understood to be the above functions U and V. Necessary and sufficient conditions are derived for this line element to have the Gaussian form, that is, the coefficient of $d\xi^2$, $d\xi d\eta$, $d\eta^2$, not to depend on $d\eta/d\xi$.

Instead of ordinary line coordinates u, v, the distance p of a line from the origin and the angle α of its normal with a fixed direction are introduced, and the forms of the line elements corresponding to the above $d\sigma^2$ in the Euclidean, elliptic, and hyperbolic planes are derived. If no two different curves of the system have a common tangent, then $d\sigma^2$ (in any form) may be used as metric for the lineal elements of the plane. In that case the radius of curvature of a curve of the system at a point becomes a single-valued function of the lineal element, which is evaluated in the Euclidean and non-Euclidean cases. An extension to three-parameter families of curves in a three-dimensional Riemann space is indicated.

H. Busemann (Los Angeles, Calif.).

Kadeřávek, František. Sur les surfaces du 4° formées par la sommation de deux surfaces du 2°. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přirodověd. **1949**, no. 14, 6 pp. (1950). (Czech. French summary)

Étant données deux droites $O\perp\Omega_\infty$ et les deux surfaces ϵ_1 et ϵ_2, on peut mener une droite P qui coupe les O, Ω, ϵ_1, ϵ_2; si o, ω, e_1, et e_2 sont les points d'intersection de la droite P avec les objects nommés, on peut chercher le lieu géométrique ϵ rempli par les points e pour lesquels $oe=(oe_2-oe_1)/2$. Dans cet article on a étudié la surface ϵ du 4°, formée par la manière formulée en supposant que $\epsilon_1=\epsilon_2$ est une surface du 2°.

Author's summary.

Müller, Hans Robert. Über eine infinitesimale kinematische Abbildung. Monatsh. Math. **54**, 108–129 (1950).

Das Studysche Übertragungsprinzip besteht in der Abbildung der gerichteten Geraden des Euklidischen Raumes auf die dualen Punkten der Einheitskugel. Einem Strahl $\mathfrak{A}$ festgelegt durch den Einheitsvektor $\mathfrak{a}$ und das Moment $\mathfrak{a}_1$ in Bezug auf O wird der duale Einheitsvektor $\mathfrak{a}+\epsilon\mathfrak{a}_1$ zugeordnet ($\epsilon^2=0$, $\mathfrak{a}^2=1$, $a\bar{a}=0$). Der Verf. ordnet nun dem Strahl $\mathfrak{A}$ das infinitesimal benachbarte Punktepaar $\mathfrak{a}$ und $\mathfrak{a}+\epsilon\mathfrak{a}_1$ der Einheitskugel zu, wobei ϵ als eine kleine Konstante betrachtet wird deren höhere als erste Potenzen zu vernachlässigen sind. Zwei Punktepaare haben zwei sphärische Abstände, welche mit dem Winkel und dem kürzesten Abstand der Geraden zusammenhängen. Die Abstände sind gleich (und das Bildpaar "isometrisch") für schneidende Gerade. Die Bildpunktepaare der Strahlen eines Büschels wandern auf zwei benachbarten Grösstkreisen, die isometrisch aufeinander bezogen sind. Untersucht werden die Abbildung der Regelscharen einer Quadrik, des Strahlbündels, des Strahlnetzes und linearen Komplexes, u.a. Umgekehrt wird untersucht welche Strahlensysteme mit gewissen infinitesimalen Transformationen der Kugel korrespondieren.

O. Bottema (Delft).

Petkantschin, B. Über die Differentialgeometrie der holomorphen Regelscharen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **40**, 261–350 (1944); **41**, 1–30 (1945). (Bulgarian. German summary)

"In der vorliegenden Arbeit wird eine vollständige Einteilung der holomorphen Regelscharen im dreidimensionalen komplexen Euklidischen Raum E_3 gegeben, wobei als Ausgangspunkt die Auffindung eines mit der Regelschar invariant verbundenen orthogonalen Dreibeins dient."

From the author's summary.

Bol, G. Zur Projektiven Differentialgeometrie der Regelflächen. Math. Z. **52**, 791–809 (1950).

The purpose of this paper is to study the projective properties of nondevelopable ruled surfaces by means of a

certain hyperbolic representation of pairs of curves on the surface. Let the equation of a conic be written in the form $2x_0x_2-x_1^2=0$. The tangents to the conic at the points $p_1(\alpha_1^2, -2^{\frac{1}{2}}\alpha_1, 1)$, $p_2(\alpha_2^2, -2^{\frac{1}{2}}\alpha_2, 1)$ intersect in a point $X(\alpha_1\alpha_2, 2^{-\frac{1}{2}}(\alpha_1+\alpha_2), 1)$. The join of corresponding points q_1, q_2 of two curves whose equations are $q^i=q_1^i(v)$, $q_2^i=q_2^i(v)$, $i=0, 1, 2, 3$, generates a ruled surface. Any point on the generator $\bar{q}$ has coordinates determined by $\bar{q}=q_1+uq_2$. Two curves on the surface are generated by the points $\bar{q}_1=q_1+\alpha_1(v)q_2$, $\bar{q}_2=q_1+\alpha_2(v)q_2$. Each of the pairs of points $\bar{q}_1$, $\bar{q}_2$ determined by $\alpha_1(v)$, $\alpha_2(v)$ is associated with the point X described above with the same values of α_1, α_2 for a given v, that is, for a given generator. The locus of X is called the hyperbolic representation of the pair of curves generated by $\bar{q}_1$, $\bar{q}_2$. The study is made by choosing the curves generated by q_1, q_2 in various manners, and by certain normalizing procedures. In particular, the curves generated by q_1, q_2 may be chosen as asymptotic curves, or flecnode curves. *V. G. Grove* (East Lansing, Mich.).

Ram Behari, and Mishra, Ratan Shanker. On the congruences of Ribaucour. Proc. Indian Acad. Sci., Sect. A. **28**, 132–141 (1948).

Two surfaces S_1 and S are said to correspond with orthogonality of linear elements if the two surfaces are in one-to-one correspondence in such a manner that the tangents to corresponding curves on S_1, S are perpendicular. If through the points x_1 of S_1 lines n be drawn parallel to the normals N to S at the corresponding point x, the locus of n constitutes a congruence of Ribaucour. The surface S_1 is called the reference surface and S the director surface of the congruence. The purpose of this paper is to add some results to those already known about such congruences. Among these new results are the following. The developable surfaces, surfaces of distribution and characteristic surfaces of a congruence of Ribaucour correspond respectively to the asymptotic curves, the lines of curvature, and characteristic lines on the director surface S. A necessary and sufficient condition that a congruence of Ribaucour be a normal congruence is that the surfaces whose spherical representations are minimal lines correspond to the conjugate systems at the corresponding points of the director surface. Moreover, the spherical representations of the lines of curvature on the director surface of a normal congruence of Ribaucour are isothermally orthogonal and isothermally conjugate. The director surface is a minimal surface. The classical method is used in the derivation of the results.

V. G. Grove (East Lansing, Mich.).

Backes, F. Sur un certain couple de surfaces projectivement applicables. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **24**, no. 1603, 29 pp. (1950).

Two surfaces S_1, S_2 are said to be projectively applicable if they may be put in one-to-one point correspondence in such a manner that their Čech-Fubini elements of projective arc lengths are identical. The purpose of this paper is to prove the existence of and to study pairs of projectively applicable surfaces such that the tangents to the curves of each of the families of asymptotic curves on S_1 intersect the tangents to the asymptotic curves of the opposite family of S_2. The two asymptotic nets are said to "cut oppositely." One of the theorems proved may be stated as follows. Let N_1, N_2 be the asymptotic nets on two surfaces S_1, S_2 which cut oppositely. If the sustaining surfaces S_1, S_2 have one of the three following properties they have the other two: (a) The surfaces S_1, S_2 are projectively applicable; (b) the lines joining corresponding points of S_1, S_2 form a W congruence, the asymptotic curves on whose focal surfaces correspond to those on S_1 and S_2; (c) the lines of intersection of corresponding tangent planes of S_1, S_2 form a W congruence on whose focal surfaces the asymptotic curves correspond to those of S_1, S_2. The method of study is that of Čech-Fubini and the method of moving frames of reference of Cartan. *V. G. Grove* (East Lansing, Mich.).

Korovin, V. I. The stratification of pairs of complexes of two-dimensional planes in five-dimensional projective space. Doklady Akad. Nauk SSSR (N.S.) **72**, 837–840 (1950). (Russian)

The two-dimensional planes in Π_5 are taken as the planes $P(A_1, A_2, A_3)$ and $P(A_4, A_5, A_6)$ of a reference frame with basic points A_i, $i, k=1, \cdots, 6$. We consider a complex (three-parametric family) of planes $P(A_1, A_2, A_3)$. We have $dA_i=\omega_i^kA_k$ and by suitable choice of A_4, A_5, A_6: $\omega_1^5=a_2\omega_3^5$, $\omega_2^6=a_3\omega_3^6$, $\omega_1^6=b_2\omega_2^5$, $\omega_3^4=a_1\omega_1^4$, $\omega_2^4=b_3\omega_3^6$, $\omega_3^5=b_1\omega_1^4$. We write $M=A_1+\lambda A_2+\mu A_3$ for a point of $P(A_1, A_2, A_3)$, where λ and μ are functions of (u, v, w). Every point M describes a three-dimensional manifold in the given Π_5. If λ and μ are, besides, considered as functions of two parameters, then we have a two-dimensional family of three-dimensional manifolds.

Two complexes P and P' of planes are called stratified if to the complexes P and P' we can adjoin two parametric families of three-dimensional manifolds Σ and Σ' such that the tangent three-dimensional plane of Σ at the point M of P passes through the corresponding plane P' and vice versa. The conditions are:

$$d\lambda=\lambda(\omega_1^1-\omega_2^2)-\omega_1^2-\mu\omega_3^2+\lambda^2\omega_2^1+\lambda\mu\omega_3^1,$$
$$d\mu=\mu(\omega_1^1-\omega_3^3)-\omega_1^3-\lambda\omega_2^3+\mu^2\omega_3^1+\lambda\mu\omega_2^1.$$

The conditions for total integrability of these equations and the analogue relations for Σ' give a system of 16 exterior quadratic equations for the pairs of complexes (P, P'). If all a_i, b_i are zero, this system can be reduced to $\omega_4^1=\omega_4^2=\omega_4^3=\omega_5^1=\omega_5^2=\omega_5^3=\omega_6^1=\omega_6^2=\omega_6^3$. This problem reduces to a system of equations in involution, of which the solution depends on 15 arbitrary functions of one variable. Some geometrical results follow. We mention the following. The complex of two-dimensional planes $P(A_1, A_2, A_3)$ can be decomposed into one parametric focal families. The locus of the foci of the decomposed focal families form in the plane P a curve of the third order. *D. J. Struik.*

Rembs, Johanna. Neue Biegungsflächen des verlängerten Rotationsellipsoids. Math. Nachr. **3**, 152–175 (1950).

The purpose of the author is to extend the results of two papers by her father [E. Rembs, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1927**, no. 5; Jber. Deutsch. Math. Verein. **39**, 278–283 (1930)] by exhibiting the equations of such surfaces, their fundamental coefficients, and discussing their singularities. Among the results may be stated the following theorem. Every prolate spheroid may be bent in such a manner that it always remains regular except for an arc of its equator which bends into a straight line whose length in the bending varies continuously from zero to the half equatorial circumference. *V. G. Grove.*

Alardin, F. Sur les surfaces représentatives des fonctions harmoniques. Bull. Soc. Math. Belgique **2** (1948–1949), 36–40 (1950).

Haag [Bull. Sci. Math. (2) **65**, 100–103 (1941); these Rev. **7**, 77] studied the asymptotic nets on surfaces whose equa-

tions are of the form $z=f(x, y)$, $f(x, y)$ being an harmonic function. The present paper studies the same type of surface S, emphasis being placed on congruences associated with the surface. In particular, if from the generic point P of S a perpendicular be dropped to the (x, y)-plane, and through the foot of the perpendicular a line l be drawn parallel to the normal to S at P, then l generates a congruence having the (x, y)-plane as a middle surface. Conversely, if there exists a plane π such that the parallels to the normals to a surface S through the feet of the projections of the points of S on π generate a congruence having π as middle surface, then, if π be made the (x, y)-plane of a coordinate system and the equation of S be written in the Monge form $z=f(x, y)$, $f(x, y)$ is harmonic.

Let $g(x, y)$ be the conjugate harmonic function of the harmonic function $f(x, y)$. Let P and P' be points on the surfaces S, S' whose equations are $z=f(x, y)$, $z=g(x, y)$, P and P' lying on a line parallel to the z-axis. Let S be a line perpendicular to the normals to S and S' at P and P'. The line Δ parallel to S through a fixed point A intersects the (x, y)-plane in a point $Q'(x, y)$. The line PP' intersects that plane in the point $Q(x, y)$. The point correspondence thus established between the points Q, Q' is directly conformal. The lines Δ generate a congruence of Ribaucour having a nonplane middle surface. These notions are finally applied to the problem of the deformation of paraboloids of revolution in the small. *V. G. Grove.*

Dubnov, J. Les réseaux sans détours. Učenye Zapiski Moskov. Gos. Univ. **100**, Matematika, Tom I, 212–216 (1946). (Russian. French summary)

[A translation of the Russian title is "Nets of equal paths on a surface."] An oriented net is said to be one of equal paths if the length of the journey from an arbitrary point M_1 to a second arbitrary point M_2 is independent of the route, with the understanding that the route follows segments of the curves of the net and any oriented segment traversed negatively contributes negative distance. This problem for plane nets was first proposed by G. Scheffer [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. **57**, 353–359 (1905)]. Representing the net by $\varphi_{\alpha\beta}du^\alpha du^\beta=0$ and the metric of the surface by $g_{\alpha\beta}du^\alpha du^\beta$, a necessary and sufficient condition that the net be one of equal paths is given along with its geometrical interpretation. In particular, there exist ∞^1 geodesic nets of this type on and only on a surface of revolution. *T. C. Doyle* (Hanover, N. H.).

***Yano, Kentarô. Setsuzoku no kikagaku. [Geometry of Connections].** Kawade shobô, Tokyo, 1947. 2+4+2+185 pp. 95 Yen.

The author says in the preface that the chief purpose of this book is to explain E. Cartan's geometrical interpretation of the concept "connection," i.e., "parallelism in the widest sense" [cf. La théorie des groupes finis et continus et la géométrie différentielle traitées par la méthode du repère mobile, Gauthier-Villars, Paris, 1937], as plainly as possible and to make clear its relations to other new differential geometries. The book is written very clearly and simply and avoids concepts that may be difficult for beginners to understand. It is not intended as a treatise for the expert, but the reviewer believes it to be one of the best for the beginner in the geometry of connections.

The book consists of 5 chapters. In order to make clear F. Klein's idea of classical geometry based on a transformation group, chapter 1 touches on group theory and then gives outlines of the Euclidean, affine, projective, and conformal geometries in a plane from the standpoint of group theory. Chapter 2 is a brief explanation of Riemannian geometry and relations between general relativity and Riemannian geometry. The main part of the book is chapter 3, which contains geometrical interpretations of Levi-Civita's parallelism [Rend. Circ. Mat. Palermo **42**, 173–205 (1917)] in Euclidean space, and of Weyl's parallelism [Raum, Zeit, Materie, $\cdots$, 3rd ed., Berlin, 1920]. There are also illustrations of Cartan's view where this parallelism is considered as Euclidean connection. In chapter 4 it is shown that there are many geometries of connections arising from the concept "connection"; the Weyl-Eddington connection and Cartan's generalized connection are exhibited briefly but clearly, and there is an exposition of three famous examples, affine, projective, and conformal connections. Chapter 5 treats the holonomy group and shows a close relationship between Klein's and Cartan's views. Only the definition and fundamental properties of the holonomy group are given, except for an example that in a space of Euclidean connection there always exists a family of ∞^1 totally geodesic hypersurfaces all of whose transversals are geodesics when a direction remains unaltered under the holonomy group of the space, and conversely.
A. Kawaguchi (Sapporo).

Hlavatý, V. Affine embedding theory. II. Frenet formulae. Nederl. Akad. Wetensch., Proc. **52**, 714–724 = Indagationes Math. **11**, 244–254 (1949).

This paper is a continuation of a previous paper [same Proc. **52**, 505–517 = Indagationes Math. **11**, 165–177 (1949); these Rev. **11**, 54] and contains a determination of two sets of η-tensors K and L, in terms of which the Frenet formulas for an A_m in A_n are derived. By these formulas the contact of two subspaces in A_n is also investigated.
C. C. Hsiung (Cambridge, Mass.).

Hlavatý, V. Affine embedding theory. III. Integrability conditions. Nederl. Akad. Wetensch., Proc. **52**, 977–986 = Indagationes Math. **11**, 356–365 (1949).

By introducing condensed indices the author first simplifies the Frenet formulas for an A_m in A_n obtained in paper II [cf. the preceding review], and then derives the integrability conditions for these formulas. The consequences of these conditions for the embedding theory as well as for the external contact invariants are studied. *C. C. Hsiung.*

Hlavatý, V. Projective geometrization of a system of partial differential equations. I. Normal points. Nederl. Akad. Wetensch., Proc. **53**, 318–326 = Indagationes Math. **12**, 66–74 (1950).

Hlavatý, V. Projective geometrization of a system of partial differential equations. II. Frenet formulae for normal points. Nederl. Akad. Wetensch., Proc. **53**, 487–493 = Indagationes Math. **12**, 139–145 (1950).

Hlavatý, V. Projective geometrization of a system of partial differential equations. III. Projective normal spaces. Nederl. Akad. Wetensch., Proc. **53**, 835–847 = Indagationes Math. **12**, 279–291 (1950).

Hlavatý, V. Projective geometrization of a system of partial differential equations. IV. Frenet formulae for projective normal spaces and their integrability conditions. Nederl. Akad. Wetensch., Proc. **53**, 848–856 = Indagationes Math. **12**, 292–300 (1950).

This sequence of four papers is concerned with a study of a projective geometrization of a system of partial differential equations, which leads to a projective embedding

theory of an X_m in a projective linear space P_n ($1 \leqq m < n$, $n \geqq 2$). The results are based partly on the affine embedding theory as developed in the author's previous papers [cf. same Proc. 52, 505–517 = Indagationes Math. 11, 165–177 (1949); these Rev. 11, 54; also the two preceding reviews]. Throughout these papers the homogeneous projective parameters for X_m are used. In paper I some fundamental transformation laws are found for constructing a connection as well as higher connections. The first connection relates X_m to a projective curved space $\mathfrak{P}_m$, while the latter ones lead to a well-defined set of "projective normal points" of the $\mathfrak{P}_m$. It is remarked that the device used in this paper may easily be generalized for an X_m in $\mathfrak{P}_n$. Paper II contains a determination of two sets of projective tensors K and L, in terms of which the Frenet formulas for the projective normal points of a $\mathfrak{P}_m$ in a P_n are derived. The formal structures of the projective tensors K and L are respectively the same as that of the affine tensors K and L defined in the author's earlier paper [see the second preceding review]. Finally, it is shown that the projective normal points are not appropriate to define projective normal spaces. In paper III associated with a point x of a "symmetric" $\mathfrak{P}_m$ the author finds a set of "privileged" points which together with x may be used to define the projective normal spaces of $\mathfrak{P}_m$ at x. In paper IV the Frenet formulas and their integrability conditions for the projective normal spaces of a symmetric $\mathfrak{P}_m$, defined in paper III, are obtained and discussed.

C. C. Hsiung (Cambridge, Mass.).

De Donder, Th. Sur les théories unitaires. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 285–289 (1950).

An invariant involving the second derivatives of the coefficients of a general bilinear form is given. It is stated that this invariant is the generalisation of the scalar Gaussian curvature of a Riemannian space.

A. H. Taub.

Walker, A. G. On Ruse's spaces of recurrent curvature. Proc. London Math. Soc. (2) 52, 36–64 (1950).

$R_{hijk,l} = R_{hijk}\kappa_l$ ($\kappa_l \neq 0$) is the defining property of a Riemannian space of recurrent curvature (K-space). Ruse [J. London Math. Soc. 21 (1946), 243–247 (1947); same Proc. (2) 50, 317–329 (1948); 50, 438–446 (1949); these Rev. 9, 102; 10, 266, 571] has introduced this notion and studied it in the cases $n = 2, 3, 4$. The author here takes up the case of general n. He finds it desirable to enlarge slightly the class of K-spaces to a class K^* by adding a restricted category of symmetric spaces. He describes two distinct subclasses of K^*, the simple spaces carrying $n-2$ parallel vector fields, and the nonsimple which always exist for $n \geqq 4$. A metric of the general nonsimple space is found and its properties investigated. Special attention is given to the K^*-spaces which are, respectively, Einstein, conformally flat, and harmonic.

J. L. Vanderslice (College Park, Md.).

Kuiper, N. H. On compact conformally Euclidean spaces of dimension >2. Ann. of Math. (2) 52, 478–490 (1950).

This paper deals with the classification of compact manifolds of dimension greater than 2, of class C^3, and with an assigned conformal structure which is locally conformally Euclidean. The main results are expressed in the following theorems. (1) A conformally complete, conformally Euclidean space of class C^3, of dimension greater than 2, admits a topological conformal mapping onto some (compact) Riemannian space of constant positive curvature. (2) A complete (in the metric sense) conformally Euclidean Riemannian space with sectional curvature greater than $k > 0$, of dimension greater than 2, of class C^3, admits a topological conformal mapping onto some (compact) Riemannian space of constant positive curvature. (3) The universal covering space of a compact conformally Euclidean space of dimension $n > 2$, of class C^3, with an infinite Abelian fundamental group, admits a conformal topological mapping onto a Euclidean hypersphere minus one or two points. The results of an earlier paper [same Ann. (2) 50, 916–924 (1949); these Rev. 11, 133] are used and some examples are given.

J. A. Schouten (Epe).

Longo, Carmelo. Trasformazioni puntuali nell'intorno di un punto unito. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 320–325 (1950).

H. Poincaré [J. Math. Pures Appl. (4) 2, 151–217 (1886)] and S. Lattès [Ann. Mat. Pura Appl. (3) 13, 1–138 (1906)] found two topological invariants of a point transformation of a plane depending on neighborhoods of the first order, and under certain conditions, two invariant curves and a reduction of the transformation to canonical form. The author first finds the topological significance of these two invariants. Then he proceeds to higher order neighborhoods with restriction to the projective group using methods developed by Bompiani and Villa for inter-plane correspondences but modified to fit the present problem. Although in the two plane case existence of invariants and canonical forms requires neighborhoods of third order at least, here in the intra-plane case second order neighborhoods are generally enough.

J. L. Vanderslice (College Park, Md.).

Bompiani, Enrico. Topologia differenziale. IV. Teoremi topologici e proiettivi sulle calotte superficiali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 169–175 (1950).

Continuing his series of papers on differential topology [cf. same vol., 3–8, 8–15, 81–86 (1950); these Rev. 11, 689] the author now studies two surface calottes of order s having a common calotte of order $s-1$. After obtaining a general theorem of a topological nature regarding the possibility of their having in common curve elements of order s, he devotes the rest of the paper to deducing further consequences for the special case of second order surface calottes in projective n-space.

J. L. Vanderslice.

Bompiani, Enrico. Topologia differenziale. V. Geometria delle superficie in uno spazio proiettivo curvo a tre dimensioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 271–275 (1950).

The author applies some of the topological notions introduced in the preceding notes of this series [see the preceding review and references cited there] to initiate the study of surfaces of a 3-dimensional projective space of paths along lines resembling the classical projective surface theory of Fubini. Specifically, there appear the asymptotic lines, the partial differential equations of the surface, and two invariant differential forms. The notion of axial system of curves is generalized and used to define projective applicability. Certain classical properties of such applicability still subsist.

J. L. Vanderslice (College Park, Md.).

Bochner, S. Vector fields on complex and real manifolds. Ann. of Math. (2) 52, 642–649 (1950).

In this paper the author shows that the existence of a certain type of family of vector fields on a compact manifold, complex or real, implies the nonexistence of a dual type of vector field. For example, if on a compact complex (real

Riemannian) V_{2k} (M_k) there exist $k+1$ covariant analytic (harmonic) vector fields $(\xi)^{(j)}$ such that for no system of $k+1$ constants $(c)\neq(0)$ the determinant formed by $(\xi)^{(j)}$ and (c) vanishes identically, then there exists on V_{2k} (M_k) no contravariant analytic (covariant Killing) vector field other than zero. *S. B. Myers* (Ann Arbor, Mich.).

Iwamoto, H. On the geometry in a space based on the notion of area. I. Tensor 9, 7–12 (1949). (Japanese)
Iwamoto, H. On the geometry in a space based on the notion of area. II. Tensor 9, 13–17 (1949). (Japanese)

The purpose of the first paper is (1) to introduce a connection into an n-dimensional space in which k-dimensional area of a k-dimensional surface: $x^i=x^i(u^\lambda)$, $i=1,\cdots,n$, $\lambda=1,\cdots,k$, is given a priori by the integral

$$O=\int_{(k)}L(x^i,p_\lambda{}^i)du^1\cdots du^k,$$

where $p_\lambda{}^i=\partial x^i/\partial u^\lambda$, and (2) to find the maximal bound of spaces of the metric class. In a space of the metric class the metric tensor $g_{ij}(x^i,p_\lambda{}^i)$ can be found from the function L and consequently a Euclidean connection can be introduced. It is proved in § 1 that in a Riemannian space the tensor $g_{i_1\cdots i_k,j_1\cdots j_k}=k!g_{[i_1|j_1|}\cdots g_{i_k]j_k}$ is expressed in a multinomial form of L, $L_i{}^\lambda$ $(=\partial L/\partial p_\lambda{}^i)$, $L_{ij}{}^{\lambda\mu}$ $(=\partial^2L/\partial p_\lambda{}^i\partial p_\mu{}^j)$. This expression enables us to define $g_{i_1\cdots i_k,j_1\cdots j_k}$ also in the most general case. In § 2 the author says without proof that the connection parameters $\omega^{i_1\cdots i_k}_{j_1\cdots j_k}(d)$ can be easily found by the condition $\delta g_{i_1\cdots i_k,j_1\cdots j_k}=0$, but this is not right and the connection parameters cannot be determined from this condition only. If the metric tensor g_{ij} exists, its uniqueness is proved in § 3, and a Euclidean connection is introduced into this space. On a k-dimensional surface the induced connection and curvature theory are also explained (in § 4). It is shown in § 5 that a space of the metric class is characterized by existence of the metric tensor g_{ij} satisfying the conditions: (I) $L^2=|g_{\lambda\mu}|$, where $g_{\lambda\mu}=p_\lambda{}^ip_\mu{}^jg_{ij}$, (II) $L^{-1}L_i{}^\lambda=g_{ij}g^{\lambda\mu}p_\mu{}^j$, (III) $(\partial g_{ij}/\partial p_\nu{}^k)p_\lambda{}^j(\delta_h{}^i-g_{hl}g^{\mu\nu}p_\mu{}^lp_\nu{}^i)=0$; and in § 6 that the metric tensor g_{ij} can be uniquely determined by the following conditions from the functions $g_{\lambda\mu}(x^i,p_\nu{}^i)$ which are given a priori as the metric tensor components of any k-dimensional surface [see A. Kawaguchi, Monatsh. Math. Phys. 43, 289–297 (1936); J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 9, 153–188 (1940); these Rev. 3, 20]: (I) $g_{\lambda\mu}=p_\lambda{}^ip_\mu{}^jg_{ij}$, (II) $L^{-1}L_i{}^\lambda=g_{ij}g^{\lambda\mu}p_\mu{}^j$, where $L=|g_{\lambda\mu}|^{\frac12}$, (III) $(\delta_i{}^h-L^{-1}L_i{}^\nu p_\nu{}^h)p_\lambda{}^k(\partial g_{hk}/\partial p_\lambda{}^j)=0$. In fact, we can find $kg_{ij}=g_{\lambda\mu}(L^{-1}L_{ij}^{\lambda\mu}+kL^{-2}L_i{}^\lambda L_j{}^\mu)$.

In the second paper the author deals with geometrization of the k-ple integral

$$O=\int_{(k)}F(x^i,\partial x^i/\partial u^\lambda,\cdots,\partial^Mx^i/(\partial u^{\lambda_1}\cdots\partial u^{\lambda_M}))du^1\cdots du^k$$

as a generalization of the first paper. He derives the intrinsic connection parameters, base connections, and the metric tensor from the function F. It is very difficult to judge whether his results are right or not, because of the brevity of this paper and the many unproved statements.

A. Kawaguchi (Sapporo).

Kawaguchi, A. Connection parameters of areal spaces. Tensor 9, 38–40 (1949). (Japanese)

The purpose of this paper is to find the concrete forms of the connection parameters in an n-dimensional space based on the notion of m-dimensional area

$$\int_{(m)}F(x^i,p_\alpha{}^i)du^1\cdots du^m,$$

where $p_\alpha{}^i=\partial x^i/\partial u^\alpha$, when the space may be not of the metric class. Using the tensor $g_{i_1\cdots i_m,j_1\cdots j_m}$ of H. Iwamoto [see the preceding review], a tensor $G^{\alpha\beta}_{ij}$ is derived which becomes $F^{2/m}g_{ij}g^{\alpha\beta}$ if the space is of the metric class, where $g^{\alpha\beta}$ is component of the metric tensor g_{ij} on the m-dimensional plane element $(x^i,p_\alpha{}^i)$. Then the coefficients Γ^*, C, B of the absolute differential $\delta v^i=dv^i+\Gamma^{*i}_{jk}v^jdx^k+C^{i\alpha}_{jk}v^j\delta p_\alpha{}^k$ and of the base connection $\delta p_\alpha{}^i=dp_\alpha{}^i+B^i{}_{\alpha k}dx^k$ are determined concretely from the postulates (I) $\delta G^{\alpha\beta}_{ij}=0$ for unaltered parameters u^α; (II) $C^{h\gamma}_{ik}G^{\alpha\beta}_{hj}=C^{h\gamma}_{jk}G^{\alpha\beta}_{hi}$; (III) $\Gamma^{*h}_{ik}=\Gamma^{*h}_{ki}$; (IV) $B^i{}_{\alpha k}=\Gamma^{*i}_{lk}p_\alpha{}^l$. It is shown that when the space is of the metric class, then these coefficients become those considered by many authors [A. Kawaguchi, Proc. Imp. Acad. Tokyo 16, 320–325 (1940); E. T. Davies, J. London Math. Soc. 20, 163–170 (1945); R. Debever, Thesis, Université Libre de Bruxelles, 1947; these Rev. 2, 167; 8, 96; 9, 379]. *A. Kawaguchi* (Sapporo).

Katsurada, Y. Generalized Gauss-Bonnet theorem. Tensor 9, 30–37 (1949). (Japanese)

In an n-dimensional space endowed with the integral $O=\int_{(m)}l(x^i,\partial x^i/\partial u^\alpha)d(u^1\cdots u^m)$ as area on an m-dimensional surface, the author gives in § 1 a geometrical meaning to the m-vector $G_{i_1\cdots i_m}$ composed of m vectors $\partial l/\partial p_\alpha{}^i$, where $p_\alpha{}^i=\partial x^i/\partial u^\alpha$. From this meaning we see that at any space element $(x^i,p_\alpha{}^i)$ this m-vector $G_{i_1\cdots i_m}$ gives the $(n-m)$-direction which may be understood to be transversal to the m-vector $p^{i_1\cdots i_m}$ composed of the m vectors $p_\alpha{}^i$ of the element. The author then generalizes the Gauss-Bonnet theorem on a hypersurface V_{n-1} in a Riemannian space V_n [Duschek, Math. Z. 40, 279–291 (1935)] to a subspace V_m in V_n (in § 2) and to a two-dimensional surface in an n-dimensional space based on the notion of two-dimensional area which is stated above (in § 3). *A. Kawaguchi.*

Moór, Arthur. Espaces métriques dont le scalaire de courbure est constant. Bull. Sci. Math. (2) 74, 13–32 (1950).

The purpose of this paper is to study Finsler spaces with constant scalar curvature which may be considered as generalizations of Riemannian spaces with constant curvature. However, the author deals only in spaces of two dimensions except for the last paragraph, and throughout the paper he uses results of Berwald on two-dimensional Finsler spaces [J. Reine Angew. Math. 156, 191–210, 211–222 (1927); Mathematica, Timişoara 17, 34–58 (1941); Ann. of Math. (2) 42, 84–112 (1941); these Rev. 3, 311; 2, 304]. After introducing the scalar curvature $\mathfrak{R}$ defined by Berwald and deriving several formulas for $\mathfrak{R}$, he proves that the relation $\mathfrak{R}=$constant is equivalent to any one of the three conditions: $\mathfrak{R}_{/r}=0$, $R_0{}^i{}_{jk/r}=0$, and $R_0{}^i{}_{0k/r}=0$, where $_{/r}$ denotes covariant derivatives and $R_0{}^i{}_{jk}=R_h{}^i{}_{jk}l^h$, where $R_h{}^i{}_{jk}$ is the Riemann curvature tensor of the Finsler space. It is shown, moreover, that if the scalar of asymmetry vanishes, then $\mathfrak{R}=\mathfrak{R}(x,y)$, i.e., $\mathfrak{R}$ is a function of position and not of direction, and conversely. In the case $\mathfrak{R}=\mathfrak{R}(x,y)$, the curvature tensor $K_{ji,kh}=l_jR_{0ikh}+R_{0ikh///j}$ is equal to $\mathfrak{R}(g_{ik}g_{jh}-g_{ki}g_{jh})$ as similar as the Riemannian space with constant curvature. The Riemann curvature tensor $R_{ij,kh}$ of a Finsler space satisfies, however, an analogous relation when $\mathfrak{R}=0$, i.e., $R^i{}_{0jk}=0$ [O. Varga, Comment. Math. Helv. 19, 367–380 (1947); these Rev. 8, 533], but not necessarily when $\mathfrak{R}$ is a nonzero constant. In a two-dimensional Finsler space there is another invariant named the principal scalar which is related to the scalar curvature by the Bianchi identity. By using the Bianchi identity the author determines the form of the principal scalar in a Finsler space with constant

scalar curvature and finds a geometric condition for $\mathfrak{R}=$ constant. There is also given the explicit form of the fundamental function $F(x, y, \dot{x}, \dot{y})$ of the Finsler space with constant curvature along an extremal in a polar coordinate system of Kneser [see p. 51 of the second cited paper of Berwald]. In the last paragraph a new scalar $(1/(n-1))R_0{}^h{}_{0h}$ is introduced in a space of n dimensions, and if this scalar is constant, then $\mathfrak{R}$ is equal to this scalar.
A. Kawaguchi (Sapporo).

***Cartan, Élie. Le calcul différentiel absolu et les problèmes récents de géométrie riemannienne.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 443–461, Rome, 1943.

This paper gives a survey of the theory of affinors and spinors in Riemannian space from an essentially geometric point of view. After an introduction concerning groups and representations, chapter II deals with affinors and their decomposition, invariant under the orthogonal group. In chapter III spinors are introduced and for $n=2\nu+1$ the geometric interpretation of a simple spinor as an isotropic ν-vector with some kind of an (extra) orientation is discussed. Especially in chapter IV the fact is emphasized that a simple spinor can be transformed into its opposite by a rotation through 2π although a ν-vector cannot. This proves that a spinor can never be represented wholly by an affinor or any combination of affinors. Covariant differentiation of spinors is considered in chapter V. *J. A. Schouten* (Epe).

Gurevič, G. B. On some linear transformations of symmetric tensors or polyvectors. Mat. Sbornik N.S. 26(68), 463–470 (1950). (Russian)

This paper establishes conditions that a tensor A, symmetrical in k upper and k lower indices, can be written in the form

$$\text{(I)}\qquad A^{x_1x_2\cdots x_k}_{i_1i_2\cdots i_k}=H^{(x_1}_{(i_1}H^{x_2}_{i_2}\cdots H^{x_k)}_{i_k)},$$

$x_1\cdots x_k, i_1\cdots i_k=1, \cdots, n$, and also deals with the corresponding case for alternating tensors (polyvectors). The following theorems are derived. (α) Let $\overset{0}{c}, \overset{1}{c}, \cdots, \overset{k}{c}$ be symmetrical tensors of valence k, where $\overset{0}{c}$ and $\overset{k}{c}$ are linearly independent. If the tensor $c=\lambda^k\overset{0}{c}+\lambda^{k-1}\mu\overset{1}{c}+\lambda^{k-2}\mu^2\overset{2}{c}+\cdots+\mu^k\overset{k}{c}$ is simple for any scalars λ, μ the c can be written in the form $(\lambda p+\mu q)^k$, where p and q are linearly independent vectors. Here the expressions $c=a^k$, $'c=a^kb^m$ stand for

$$c_{i_1i_2\cdots i_k}=a_{i_1}\cdots a_{i_k},$$
$$'c_{(i_1i_2\cdots i_kj_1\cdots j_m)}=a_{(i_1\cdots i_k}b_{j_1\cdots j_m)}.$$

(β) Let $A^{x_1x_2\cdots x_k}_{i_1i_2\cdots i_k}$ be symmetric in the upper and in the lower indices and of rank $r\neq 1$. Then A can be written in the form (I) if and only if it transforms an arbitrary simple symmetrical covariant k-valent tensor into a simple tensor of the same rank. This is equivalent to

$$A^{x_1x_2\cdots x_k}_{i_1i_2\cdots(i_k}A^{y_1y_2\cdots y_k}_{j_1)j_2\cdots j_k}\neq 0,$$
$$A^{(x_1x_2\cdots x_k}_{i_1i_2\cdots(i_k}A^{y_1)y_2\cdots y_k}_{j_1)j_2\cdots j_k}=0.$$

(γ) Let q linearly independent k-valent polyvectors (k-vectors) be such that any linear combination of them is simple. Then either (1) all these k-vectors can be obtained from one and the same simple $(k-1)$-vector, or (2) all lie in one and the same $(k+1)$-dimensional space. Case (1) is only possible if $q\leqq n-k+1$, case (2) only if $q\leqq k+1$. (δ) Let $C_n{}^k$ linearly independent simple covariant k-vectors w^S be given in n-space, where S is a combination (without repetition) of k out of the n indices $1, \cdots, n$, and also $C_n{}^{k-1}$ simple $(k-1)$-vectors v^T, where T is a combination of $k-1$ of the same indices ($T\epsilon S$). If the w^S can be obtained from the v^T when S contains every T, then it is possible to find n linearly independent vectors $\overset{1}{p}, \cdots, \overset{n}{p}$ such that the k-vector w^S can be obtained from the vector $\overset{i}{p}$ if the combination S contains the index i ($i\epsilon S$). (ϵ) Let $A^{x_1\cdots x_k}_{i_1\cdots i_k}$ be a tensor skew symmetrical in the upper and in the lower indices and of nonsingular matrix. Let the dimension of space n be $\neq 2k$. Then the tensor A can be written in the form

$$A^{x_1x_2\cdots x_k}_{i_1i_2\cdots i_k}=H^{[x_1}_{[i_1}H^{x_2}_{i_2}\cdots H^{x_k]}_{i_k]}$$

if and only if it transforms an arbitrary simple covariant k-vector into a simple covariant k-tensor. This is equivalent to

$$A^{(\alpha_1(\alpha_2\cdots(\alpha_k}_{[i_1i_2\cdots i_k}A^{\beta_k)\cdots\beta_2)\beta_1)}_{j_1j_2]j_3\cdots j_k}=0.$$

The case $n=2k$ is also discussed. The author quotes his book [Foundations of the Theory of Algebraic Invariants, OGIZ, Moscow-Leningrad, 1948; these Rev. 11, 413].
D. J. Struik (Cambridge, Mass.).

Roussopoulos, A. Note sur le calcul tensoriel. Bull. Soc. Math. Grèce 25, 68–81 (1950).

A transformation of the components of a geometric object, induced by a transformation of coordinates, was formerly, and is still by many authors, denoted by changing or marking the kernel letter, e.g., v^κ and $\bar{v}^\kappa$, $\kappa=1, \cdots, n$. For more complicated cases this method gives rise to many difficulties, especially if quantities of a valence greater than 1 are considered and if intermediate components are used, i.e., components with respect to more than one coordinate system. These difficulties vanish if one uses the kernel-index method, leaving the kernel unchanged and changing only the indices, e.g., v^κ, $v^{\kappa'}$, $\kappa=1, \cdots, n$, $\kappa'=1', \cdots, n'$. The author is apparently unaware of this method and tries to get a similar result by introducing an extra index for every running index, indicating the coordinate system to which this running index belongs, e.g., $P^{\kappa\lambda\mu}_{\alpha\beta\gamma}$ instead of $P^{\kappa\lambda'\mu''}$ in kernel-index notation. This new method of course accomplishes all things that could be done with the kernel-index method but only at the expense of doubling the number of indices. The remark in the résumé that the notion of "tensor" is generalized is erroneous. The author deals with the same geometric objects as other authors; he considers only their intermediate components, which also has been done previously by others. *J. A. Schouten* (Epe).

NUMERICAL AND GRAPHICAL METHODS

***Table of the Bessel Functions $Y_0(z)$ and $Y_1(z)$ for Complex Arguments.** Prepared by the Computation Laboratory, National Bureau of Standards. Columbia University Press, New York, N. Y., 1950. xl+427 pp. $7.50.

This volume contains the first extensive tables of Bessel functions of the second kind for complex variable. It gives 10D values of the real and imaginary parts of $Y_n(\rho e^{i\varphi})$ for $n=0, 1$, $\rho=0(.01)10$, and $\varphi=0(5°)90°$. The Bessel functions of the first kind were tabulated in the same range in an earlier volume [Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments, Columbia University Press, New York, 1st ed., 1943, 2d ed., 1947; these Rev.

5, 159; 9, 103]; by a combination of the tables in both volumes one may obtain values of Bessel functions of the third kind, modified Bessel functions, and Kelvin's ber and bei functions. Since the Bessel functions of the second kind possess a singularity at the origin, interpolation for small ρ in the main tables of the present volume becomes difficult. For this reason the real and imaginary parts of the auxiliary functions $Y_0(\rho e^{i\varphi})-(2/\pi)J_0(\rho e^{i\varphi})\log c$ and $Y_1(\rho e^{i\varphi})-(2/\pi)J_1(\rho e^{i\varphi})\log\rho+(2/\pi\rho)e^{-i\varphi}$ are tabulated to 10D for $\rho=0(.01).50$ and $\varphi=0(5°)90°$. Further supplementary material consists of 9D tables of the first 3 complex zeros of Y_0 with values of Y_1 and Y_1' at the zeros, the first 3 complex zeros of Y_1 with values of Y_0 $(=Y_1')$, and the first 4 complex zeros of Y_1' with values of Y_0 and Y_1; a 5D table of the first 15 zeros of Y_0, Y_1, and Y_1'; a 10D table of five-point Lagrangean interpolation coefficients; and charts of contour lines of the real and imaginary parts, also of modulus and phase, of $Y_0(z)$ and $Y_1(z)$.

The foreword by von Mises explains some applications of the functions tabulated in this volume, and expresses the view that tables of special functions will prove useful for a long time to come, in spite of the change in computational techniques caused by the advent of high-speed computers. The introduction by Lowan gives an account of some properties of Bessel functions, of the method of computation of the present volume, and of the interpolation to be used in connection with it. There is a note by Hillman on complex zeros of Bessel functions [cf. also Bull. Amer. Math. Soc. 55, 198–200 (1949); Math. Tables and Other Aids to Computation 3, 351–352 (1949); these Rev. 10, 704, 740].

A. Erdélyi (Pasadena, Calif.).

Wilkes, M. V., and Renwick, W. The EDSAC (Electronic delay storage automatic calculator). Math. Tables and Other Aids to Computation 4, 61–65 (1 plate) (1950).

Wilkes, M. V. The use of the EDSAC for mathematical computation. Appl. Sci. Research B 1, 429–438 (1950).

Wheeler, D. J. Programme organization and initial orders for the EDSAC. Proc. Roy. Soc. London. Ser. A. 202, 573–589 (1950).

The mathematical laboratory of the University of Cambridge has as one of its principal tools an automatic calculating machine, called the EDSAC, which has been described by Wilkes and Renwick [cf. J. Sci. Instruments 26, 385–391 (1949); these Rev. 11, 401]. The purpose of the present paper is to set out the fundamentals of the coding for this machine. This is done quite completely. There is especial emphasis placed upon the methods used for handling subroutines. There are three appendices to the paper in which the author illustrates the principles discussed in the body of the text.

H. H. Goldstine (Princeton, N. J.).

Huskey, H. D. Characteristics of the Institute for Numerical Analysis computer. Math. Tables and Other Aids to Computation 4, 103–108 (1950).

The incorporation of subroutines into a complete problem on the NBS Eastern Automatic Computer. Math. Tables and Other Aids to Computation 4, 164–168 (1950).

Cohen, Arnold A. Magnetic drum storage for digital information processing systems. Math. Tables and Other Aids to Computation 4, 31–39 (1950).

Goldstine, H. H., and A. The electronic numerical integrator and computer (ENIAC). Gaceta Mat. (1) 2, 141–156 (1 plate) (1950). (Spanish)

Translated from Math. Tables and Other Aids to Computation 2, 97–110 (1946); these Rev. 8, 354.

Schepler, Herman C. The chronology of pi. Math. Mag. 23, 165–170, 216–228, 279–283 (1950).

The author gives a list of approximate values given for π at various times and in various places, together with related material, such as notes on notorious circle-squarers. The reviewer noticed several minor errors and omissions. Shanks's 707 decimal value is reproduced [p. 229] but only on p. 283 is it remarked that it is incorrect [cf. Ferguson and Wrench, Math. Tables and Other Aids to Computation 3, 18–19 (1948); these Rev. 9, 308]; the final entry refers to the recent computation to 2035 decimals [see the following review]. Oddly enough, neither these references, nor a series of older ones leading up to them, are given by the author.

R. P. Boas, Jr. (Evanston, Ill.).

Reitwiesner, George W. An ENIAC determination of π and e to more than 2000 decimal places. Math. Tables and Other Aids to Computation 4, 11–15 (1950).

The author gives π to 2035 decimal places and e to 2010 decimal places, and briefly describes the method of computation. An independent computation by Wrench and Smith [same vol., 160–161 (1950)] confirms the value of π to 1118 decimal places. The paper reviewed below gives 490 more places of e.

R. P. Boas, Jr. (Evanston, Ill.).

Metropolis, N. C., Reitwiesner, G., and von Neumann, J. Statistical treatment of values of first 2,000 decimal digits of e and of π calculated on the ENIAC. Math. Tables and Other Aids to Computation 4, 109–111 (1950).

The authors state that the first 2000 decimal digits of π [cf. the preceding review] seem to be random, and present calculations indicating that those of e deviate significantly from randomness, in the sense that the numbers of each digit stay too close to their expected values. This trend continues, though less strongly, up to 2500 digits.

R. P. Boas, Jr. (Evanston, Ill.).

Neĭšuler, L. Ya. On separable interpolation of certain classes of functions of several variables. Doklady Akad. Nauk SSSR (N.S.) 71, 1023–1026 (1950). (Russian)

The author considers interpolation in tables of composite functions of three variables of the form

$$f(x,y,z)=f_2(f_1(x,y),z)$$

and exhibits skeleton tables indicating the successive application of linear interpolation to the several variables. The general case of many variables is touched upon.

D. H. Lehmer (Berkeley, Calif.).

Stoyanoff, A. Quelques remarques pratiques sur l'interpolation. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 38, 217–247 (1942). (Bulgarian. French summary)

Prokop, W. Zur Graeffeschen Methode für die Auflösung algebraischer Gleichungen. Elemente der Math. 5, 115–116 (1950).

Summary of a lecture given by A. Ostrowski in February, 1950.

Bloh, Z. Š. A grapho-analytic method of estimation of the stability of linear regulating systems. Avtomatika i Telemehanika 8, 441–450 (1947). (Russian)

The stability of the system depends upon the signs of the real parts of the roots of a polynomial equation. The author presents some graphical methods for determining the location of the roots. *R. Bellman.*

Flomenhoft, H. I. A method for determining mode shapes and frequencies above the fundamental by matrix iteration. J. Appl. Mech. 17, 249–256 (1950).

Supposons connues r colonnes modales d'une matrice d'ordre n. Soit (A_i) une autre colonne modale. Les relations d'orthogonalité entre modes permettent d'écrire

$$A_i = \sum_{j=r+1}^{n} K_{ij}^{(r)} A_j, \qquad i=1, \cdots, r.$$

La méthode de Pipes [voir, par exemple, Applied Mathematics for Engineers and Physicists, McGraw-Hill, New York-London, 1946, pp. 191–203] pour déterminer une nouvelle colonne modale consiste à construire avec la matrice proposée et la matrice des $K^{(r)}$ une matrice d'ordre $n-r$ dont les colonnes modales sont les $n-r$ colonnes inconnues. L'auteur établit les formules liant les $K^{(r)}$ aux $K^{(r-1)}$. Il indique comment disposer pratiquement les calculs et comment les vérifier. *J. Kuntzmann* (Grenoble).

Scorer, R. S. Numerical evaluation of integrals of the form $I=\int_{x_1}^{x_2} f(x)e^{i\phi(x)}dx$ and the tabulation of the function $\mathrm{Gi}(z)=(1/\pi)\int_0^\infty \sin(uz+\frac{1}{3}u^3)du$. Quart. J. Mech. Appl. Math. 3, 107–112 (1950).

Integrals of the form in the title may be evaluated by making use of the principle of stationary phase which depends upon finding an x_0 satisfying $\phi'(x_0)=0$. If $f(x)$ is slowly varying near $x=x_0$ and $e^{i\phi(x)}$ is sufficiently oscillatory elsewhere, then in the neighbourhood of x_0,

$$I=f(x_0)\int_{x_1}^{x_2} \exp\{i(\phi_0+\tfrac{1}{2}(x-x_0)^2\phi_0'')\}dx.$$

Such an approximation is only valid when the succeeding terms in the Taylor expansion are negligible over a sufficient range of x, which is the case when $|\phi_0'''| \ll \{|\phi_0''|\}^{\frac{3}{2}}$ [Lamb, Hydrodynamics, 6th ed., Cambridge University Press, 1932, art. 241]. The author has devised a method to evaluate I when ϕ_0''' is not negligible, for $\phi(x)=a+bx+cx^2+dx^3$. The calculation of I is reduced to that of $K(z)$ by the asymptotic formula $I\sim\pi(3d)^{-\frac{1}{3}}f(x_0)e^{i\beta}K(z)$, where

$$\beta=a-\tfrac{1}{3}c-bc/3d+c^3/9d^2,$$

$$K(z)=\mathrm{Ai}(z)+i\mathrm{Gi}(z)=\pi^{-1}\int_0^\infty \exp\{i(uz+\tfrac{1}{3}u^3)\}du,$$

where $\mathrm{Ai}(z)$ is already tabulated and $\mathrm{Gi}(z)$ is a particular integral of $y''-zy=-\pi^{-1}$ determined by the initial conditions $\mathrm{Gi}(0)=3^{-7/6}\{\Gamma(\frac{2}{3})\}^{-1}$, $\mathrm{Gi}'(0)=3^{-5/6}\{\Gamma(\frac{1}{3})\}^{-1}$, from which $\mathrm{Gi}(z)$ is obtained by numerical integration. The functions $\mathrm{Gi}(z)$ and $\mathrm{Hi}(-z)=\pi^{-1}\int_0^\infty \exp(-uz-\frac{1}{3}u^3)du$ are tabulated in the range $z=0(.1)10$ in 7 decimals. Outside this range the asymptotic formulae

$$\mathrm{Gi}(z)\sim\frac{1}{\pi}\left(\frac{1}{z}+\frac{2!}{z^4}+\frac{5!}{3z^7}+\cdots\right),$$

$$\mathrm{Hi}(-z)\sim\frac{1}{\pi}\left(\frac{1}{z}-\frac{2!}{z^4}+\frac{5!}{3z^7}-\cdots\right)$$

give at least 9 correct decimals. *S. C. van Veen.*

Stoyanoff, A. Sur le calcul approché des intégrales définies. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 37, 499–521 (1941). (Bulgarian. French summary)

"Dans la première partie nous expliquons pourquoi la formule de Simpson donne parfois des résultats excellents, parfois des résultats très médiocres. Dans la seconde et la troisième partie nous donnons quelques méthodes pratiques pour le calcul de $\int_a^b f(x)dx$ dans le cas où la formule de Simpson donne de mauvais résultats."

From the author's summary.

Wallman, Henry. An electronic integral-transform computer and the practical solution of integral equations. J. Franklin Inst. 250, 45–61 (1950).

The author describes an electronic instrument for the graphical evaluation of the integral transform

$$F(x)=\int_a^b K(x,t)f(t)dt$$

from a given graphical input of $f(t)$. The kernel $K(x,t)$ is represented on a photographic plate by making its optical opacity at position (x,t) proportional to $K(x,t)$ which is sensed as a time variable voltage by a television process scanning 100 equidistant x-strips consecutively at the rate of 1/1000 second per strip. A graph of $f(t)$ is conveyed to a cathode ray curve follower which is timed to convert the impulses $K(x,t)$ to the products $K(x,t)f(t)$ for each of the 100 sets of x-strip impulses. These are then averaged and conveyed as 100 consecutive horizontal deflections to a long persistence cathode ray tube screen exhibiting the integral transform $F(x)$ as a graph. Since the preparation of the photographic plate for the kernel $K(x,t)$ is comparatively laborious the instrument is particularly suited to problems in which the same kernel is applied to a multitude of $f(t)$. Fourier transforms, expansions into orthogonal series, Hilbert transforms, the convolution integral, solution of the Fredholm integral equation of the 2d kind by simultaneous algebraic equations and by iterated kernels are discussed as examples. The iteration of kernels is made automatic by a "time compressed" transmission of the output graph $F(x)$ to the input curve follower as the new $f(t)$. Generalisations to nonlinear problems of the types $\int K(x,t)k(t,f(t))dt$, $\int K(x,t)k(x,f(t))dt$, $\int K(x,t)f(k(x,t))dt$ are discussed and illustrated with boundary value problems.

H. O. Hartley (London).

Milne, W. E. Note on the Runge-Kutta method. J. Research Nat. Bur. Standards 44, 549–550 (1950).

The author makes a comparison between the standard Runge-Kutta method of solving the differential equation $y'=f(x,y)$ and the step-by-step method of numerical quadrature. He notes two special advantages of the Runge-Kutta method: (1) No special devices are required for starting the computation, and (2) the length of the step can be modified at any time in the course of the computation without additional labour. However, this method is open to two serious objections: (1) The process does not contain in itself any simple means for estimating the error or for detecting computation mistakes, and (2) each step requires four substitutions into the differential equation, which may demand an excessive amount of labour per step. The author remarks that examples in textbooks which are used to illustrate the Runge-Kutta method are such that the method appears in a very favorable light. However, for the innocent-looking example $y'=5y(1+x)^{-1}$, $y(0)=1$, the Runge-Kutta

method does not make a very good showing when compared with a step-by-step method of numerical quadratures (central differences). The advantages of the step-by-step method are: (1) By internal evidence the computation shows how many places in y can be accepted as reliable, and (2) except at the beginning only one substitution is required per step. [It may be added that the errors may be corrected in a very simple way.] On the other hand, the disadvantages are: (1) Special devices are needed to get started, and (2) change of the interval h in the course of a computation is somewhat troublesome. The accuracy of the last method (shown by examples) is indeed very much greater (40 to 100 times) than that of the Runge-Kutta method. The possible occurrence of very great errors, difficult to ascertain, is a serious indictment of the Runge-Kutta method.

S. C. van Veen (Delft).

Miller, J. C. P., and Mursi, Zaki. Notes on the solution of the equation $y''-xy=f(x)$. Quart. J. Mech. Appl. Math. 3, 113–118 (1950).

A linear differential equation of the second order is reduced to the normal form (1) $d^2y/dx^2+I(x)y=F(x)$ by an appropriate choice of dependent or independent variable, or both. In the case $I(x)=\sum_{k=0}^{\infty}I_k(x-x_0)^k/k!$, by a subdivision of the full range of x, an approximation sufficient for many purposes may be found by using the first few terms of this expansion. This paper is concerned with the case where the intervals are chosen so that linear interpolation of $I(x)$ in the form $I_0+I_1(x-x_0)$ is sufficient. By writing $I_0+I_1(x-x_0)=-I_1^{\frac{1}{3}}\xi$ equation (1) is converted to the form (2) $d^2y/d\xi^2-\xi y=f(\xi)$. The general solution of $d^2y/dx^2-xy=0$ is $u=a\operatorname{Ai}(x)+b\operatorname{Bi}(x)$, where $\operatorname{Ai}(x)$ and $\operatorname{Bi}(x)$ are the Airy integral functions, and a and b are arbitrary constants. If in (2), $f(x)=g(x)u+h(x)du/dx$ ($g(x)$ and $h(x)$ power series in x), then it is shown by eliminating powers of x that $f(x)=\sum_{k=0}^{\infty}a_kd^ku/dx^k$, where the coefficients a_k are linearly expressed in the coefficients of $g(x)$ and $h(x)$; in this case, the general solution of (2) is $y=\sum_{k=0}^{\infty}a_ku_{k+1}/(k+1)$. If $f(x)$ is expressed as a power series, $f(x)=\sum_{k=0}^{\infty}b_kx^k$, then the solution of (2) is in the form $y=k(x)+l(x)v$, where $k(x)$ and $l(x)$ are power series in x, and v is the solution $\int_0^\infty \sin(ux+\frac{1}{3}u^3)du+u$ of the equation $d^2y/dx^2-xy=-1$. More generally, the solution of (1) is formally expressed by $y=\sum_{k=0}^{\infty}w_{3k}$, where $w_{3k+3}=-x^{-1}d^2w_{3k}/dx^2$ and $w_0=f(x)/x$. *S. C. van Veen.*

Olver, F. W. J. A new method for the evaluation of zeros of Bessel functions and of other solutions of second-order differential equations. Proc. Cambridge Philos. Soc. 46, 570–580 (1950).

Let $F(z)$, $G(z)$ denote independent solutions of

$$w''+p(z)w'+g(z)w=0.$$

Then $C(z,\alpha)\equiv F(z)\cos\alpha-G(z)\sin\alpha$ is also a solution. The equation $C(z,\alpha)=0$ determines z as a function of α, say $z=\rho(\alpha)$. It is shown that $\rho(\alpha)$ satisfies a certain third order nonlinear differential equation:

$$2\rho'\rho'''-3\rho''^2+\{4q(\rho)-[p(\rho)]^2-2p'(\rho)\}\rho'^4-4\rho'^2=0.$$

If this is integrated we get, for each α, the value of a zero of the corresponding $C(z,\alpha)$; in particular, we get zeros of $F(G)$ when α is an odd (even) multiple of $\frac{1}{2}\pi$. This integration has been carried out numerically in the case of Bessel functions. Here it is possible to obtain a first approximation to the solution by neglecting the second and third derivatives of ρ with respect to α. Various other devices for shortening the work are described, as are the modifications necessary for the convenient handling of the early zeros. Some details are given in the case of the computation to 10D of the first thirty zeros of J_{30}, Y_{30}, starting from the thirty-first zero of Y_{30}. These techniques have been used to supplement the usual ones, inverse interpolation, McMahon expansion, and that of Miller-Jones expansion [Bickley and Miller, Philos. Mag. (7) 36, 121–124, 124–131, 206–210 (1945); Bickley, ibid., 131–133 (1945); Miller and Jones, ibid., 200–206 (1945); these Rev. 7, 82, 83, 82], in carrying out the computational program of the Royal Society Mathematical Tables Committee. *J. Todd.*

Sobrero, Luigi. Un metodo di approssimazioni successive per la risoluzione del problema di Dirichlet. Ann. Scuola Norm. Super. Pisa (3) 3 (1949), 67–93 (1950).

Zur Lösung des Dirichlet'schen Problems, d.h. zur Lösung der Gleichung $\partial^2f/\partial x^2+\partial^2f/\partial y^2=0$ bei vorgegebenen Randwerten längs einer einen einfach zusammenhängenden Bereich umschliessenden Kurve, wird folgendes Verfahren der successiven Approximationen behandelt: Jedem Punkt P im Inneren des Bereiches wird eine möglichst grosse Kreislinie C zugeordnet, die ganz im Inneren des Bereiches verläuft. Man geht nun von einer willkürlichen, die vorgeschriebenen Randwerte annehmenden Funktion $f_0(x,y)$ aus und ordnet dem Punkt P den Mittelwert der Funktion $f_0(x,y)$ längs der Kreislinie zu. Die so erhaltene Funktion bezeichnet man mit $f_1(x,y)$ und verfährt mit dieser in analoger Weise. Das Verfahren ist nicht neu. Der Verfasser zitiert Lebesgue [C. R. Acad. Sci. Paris 154, 335–337 (1912)] und Gimmino [Rend. Sem. Mat. Univ. Padova 3, 46–66 (1932)]. Der Verfasser gibt für dieses Verfahren der successiven Approximationen einen ausführlichen Konvergenzbeweis, und zwar in der Weise, dass es ihm gelingt zu zeigen, dass der Fehler nach Durchführung der n-ten Approximation kleiner ist, als $(a+bn)^{-2}$, wobei a, b passend gewählte Konstanten sind. Zu den Funktionen $f_\nu(x,y)$ werden in geeigneter Weise Majoranten gebildet und die Durchführung des Konvergenzbeweises führt dann auf Recursionsformeln des folgenden Typus: $K_{n+1}=K_n+m_n/\rho$, $m_{n+1}=m_n(1-a(m_n/K_n)^4)$.

P. Funk (Wien).

Chu, E. L. Upper and lower bounds of eigenvalues for composite-type regions. J. Appl. Phys. 21, 454–467 (1950).

Die Arbeit enthält die Darlegung einer Methode zur Berechnung der oberen und unteren Schranke von Eigenwerten, die von Schwinger in Vorlesungen an Beispielen durchgeführt wurde. Das Problem für den kleinsten Eigenwert wird in folgender Weise gefasst: $I[u]=D[u]+\int_\Gamma p\sigma u^2ds$, mit der Nebenbedingung $\iint_R\rho u^2dx\,dy=1$, wobei

$$D[u]=\iint_R p(u_x^2+u_y^2)dx\,dy+\iint_R qu^2dx\,dy,$$

$$I[u]=D[u]-\mu\iint_R pu^2dx\,dy-2\sum_1^m c_i\int_\Gamma(u-f(s))g_i(s)\,ds.$$

Dabei soll u stetig sein und stückweise stetige erste Ableitungen in R besitzen, und σ ist eine stückweise stetige Funktion der Koordinaten des Randes. Kennzeichnend für diese Methode ist folgender Satz: Man wähle für $u(s)$ längs des Randes eine solche Funktion, sodass $u(s)=0$ auf Γ, wo $\sigma=\infty$. Ferner genüge u der partiellen Differentialgleichung $L(u)+\lambda\rho u=0$. Man denke sich $\partial u/\partial n$ längs Γ berechnet und löse hierauf die Gleichung $\int_\Gamma pu(\partial u/\partial n)ds=-\int_\Gamma p\sigma u^2ds$ für λ

auf. Der so erhaltene Wert ergibt eine obere Schranke. Ein ähnlicher Satz wird auch für die untere Schranke gezeigt. Ferner wird eine Methode angegeben, wie man bei der Bestimmung der höheren Eigenwerte auf die exakte Kenntnis der unteren Eigenfunktion verzichten kann.

P. Funk (Wien).

Kahan, Théo. Sur une méthode variationnelle dans les problèmes de diffraction et de diffusion des ondes brogliennes. C. R. Acad. Sci. Paris **230**, 2075–2077 (1950).

In ähnlicher Weise wie die Ritz'sche Methode zur Bestimmung der Eigenwerte benützt wird, wird hier ein Variationsproblem dazu verwendet, um bei der Schrödinger-Gleichung für ein System von zwei Partikeln den in der asymptotischen Lösung auftretenden Phasenwinkel zu bestimmen. *P. Funk* (Wien).

Von Neumann, J., and Richtmyer, R. D. A method for the numerical calculation of hydrodynamic shocks. J. Appl. Phys. **21**, 232–237 (1950).

The main object of this paper is a practicable numerical method for computing hydrodynamic flow involving shocks. Determination of the position of these shock discontinuities is part of the initial-boundary value problem, and this fact is a formidable obstacle to actual solution of problems. As suggested before, one could overcome the difficulty by replacing the system S of equations, which is of the first order, by another system S', of second order, which takes into account the factor λ of heat conduction and the factor μ of viscosity. For $\lambda, \mu \rightarrow 0$ the system S' tends to S, and it is plausible that the solutions of S', all of which are continuous, tend in the limit to solutions of S, exhibiting the shock discontinuities in question. This remark immediately suggests numerical procedures which, however, appear rather complicated. In the present paper the authors have modified the attempt in a remarkable way. Again the system S is replaced by another system S'', of second order and having continuous solutions; but the authors observe that the system S'' can be chosen in many ways, depending on one parameter, so that in the limit as this parameter tends to zero the system S results and that the solutions of S'' tend to the solutions of S, exhibiting the proper shocks. The specific proposal in the paper is to replace in S the pressure p by an expression $p+q$, where q, containing the first derivative of the specific volume, is properly chosen. This replacement, while not having direct physical significance, lends itself to a simpler theoretical and numerical treatment than that involving heat conduction and viscosity. The paper discusses a simple special case. Furthermore, general schemes for finite difference procedures are set up and the stability of the computational procedures concerning these finite difference schemes is ascertained. *R. Courant.*

Hidaka, Koji. Stencils for integrating partial differential equations of mathematical physics. Math. Japonicae **2**, 27–34 (1950).

This paper presents a number of computing stencils for difference expressions associated with the differential operators $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x\partial y$, $\nabla^2 f$, $\nabla^2\nabla^2 f$, and $\nabla^2\nabla^2\nabla^2 f$. The author uses a "lozenge" of points in the shape of a square with 45° sides, the side of the lozenge varying from three points to six points. Not all stencils that could be made for a given lozenge have been included.

W. E. Milne (Los Angeles, Calif.).

Dormont, Henri. Étude rhéographique des champs laplaciens à structure hélicoïdale. Philips Research Rep. **5**, 262–269 (1950).

Two different methods are indicated for determining equipotentials of a spirally wound system of conductors by means of the electrolytic tank. *Author's summary.*

Luthin, James N., and Gaskell, R. E. Numerical solutions for tile drainage of layered soils. Trans. Amer. Geophys. Union **31**, 595–602 (1950).

Bereis, R. Mechanismen zur Verwirklichung der Joukowsky-Abbildung. Österreich. Ing.-Arch. **4**, 252–256 (1950).

This paper is essentially the same as a paper of the same title by this author [Arch. Math. **2**, 126–134 (1950); these Rev. **11**, 406]. *M. Goldberg* (Washington, D. C.).

Sekiya, Tuyosi. Graphical method of composing the rotating displacement of a rigid body fixed a point only. J. Osaka Inst. Sci. Tech. Part I. **1**, 115–116 (1949).

Hubeny, K. Die konforme Kegelprojektion mit zwei längentreu abgebildeten Parallelkreisen. Österreich. Z. Vermessgswes. **37**, 126–140 (1949).

As an aid in the numerical calculation for a conformal conic projection with two true scale parallels there are developed power series expressions in terms of the meridian arc length for projection coordinates, meridian convergence, and scale distortion. *N. A. Hall* (Minneapolis, Minn.).

Skowron, Thaddeus S. Punch card method of converting geographic coordinates to universal transverse mercator grid coordinates. Trans. Amer. Geophys. Union **31**, 511–517 (1950).

Allen, William A. Affine transformations applied to the multiplex aero projector. Photogrammetric Engrg. **16**, 581–589 (1950).

Hauer, F. Über die Bestimmung der Grösse des Vermessungsbereiches der Niederen Geodäsie. Österreich. Z. Vermessgswes. **37**, 42–55 (1949).

In establishing the bounds for lower geodesy, it is suggested here that the limits of surveying accuracy be correlated with the cartographic limitations of a rectangular grid. Assuming an accuracy in length determination of 1 in 50,000 it is shown that the corresponding mapping domain for lower geodesy extends over a circle of 40km radius. Similar bounds are obtained for rectangular domains.

N. A. Hall (Minneapolis, Minn.).

ASTRONOMY

Ghosh, N. L. An extension of Hamy's theorem to rotating gaseous bodies. Bull. Calcutta Math. Soc. **41**, 92–102 (1949).

The author proved in a previous paper [same Bull. **40**, 229–230 (1948); these Rev. **10**, 746] that it is not possible for an incompressible fluid rotating mass to have a density distribution characterized by similar ellipsoids. This theory was proved to hold quite generally under any gravitational field due to external sources in addition to the self gravitation of the mass. In the present paper the author proves a similar

theory for a rotating mass where density is not necessarily constant, but may be a function of pressure. This theory is, however, only proved for the case of self gravitation without external sources. The method used is similar to the one from the earlier case. It consists in deducing a partial differential equation for the density, and introducing the density as a function of an ellipsoidal coordinate by taking into account the necessary boundary conditions. It is then shown that such solutions are impossible. *G. Randers.*

Ghosh, N. L. Corrections to my paper on 'An extension of Hamy's theorem, etc.' Bull. Calcutta Math. Soc. **41**, 220 (1949).

See the preceding review.

Mercier, Robert. Une nouvelle méthode pour le calcul des orbites des étoiles doubles visuelles. C. R. Acad. Sci. Paris **231**, 819–821 (1950).

Matukuma, T. A criterion formula for the reflection effect in close binary stars. Sci. Rep. Tôhoku Univ., Ser. I. **33**, 166–173 (1950).

The illumination of one spherical star by another is investigated. The illuminating star (of radius r_1) is assumed to radiate with an intensity which varies linearly with the cosine of the emergent angle while the surface of the illuminated star (of radius r_2) reflects the radiation it receives in accordance with Lambert's law with unit albedo. The final results are expressed as a series in $(r_1/a)^2$ and $(r_2/a)^2$, where a denotes the separation of the two stars and the first order "corrections" are explicitly evaluated. [There are numerous misprints.] *S. Chandrasekhar* (Williams Bay, Wis.).

Busbridge, Ida W. On a recent paper on radiative equilibrium by D. H. Menzel and H. K. Sen. Astrophys. J. **111**, 654–657 (1950).

Placzek and Seidel [Physical Rev. (2) **72**, 550–555 (1947); these Rev. **9**, 147] have shown that by writing $b(s)=s\int_0^\infty B(t)e^{-st}dt$ $(\Re(s)>0)$, the Schwarzschild-Milne integral equation $B(\tau)=\frac{1}{2}\int_0^\infty B(t)E_1(|t-\tau|)\,dt$ becomes $b(s)\alpha(s)=\int_0^1 b(1/\mu)\mu(1-\mu s)^{-1}d\mu$, where

$$\alpha(s)=s^{-2}\{\log[(1+s)/(1-s)]-2s\}.$$

Expressing $B(\tau)$ in the form $B(\tau)=a+b\tau+\int_0^1\phi(y)e^{-\tau/y}dy$, the author shows that $\phi(y)$ satisfies the equation

$$\left(\frac{a}{s^2}+\frac{b}{s^3}\right)\log(1+s)-\frac{a}{s}-\frac{b}{s^2}+\frac{b}{2s}$$
$$(*)\qquad +\int_0^1\frac{y^2\phi(y)}{sy+1}\log\left(1+\frac{1}{y}\right)dy$$
$$+\left\{\frac{1}{s}\log(1+s)-2\right\}\int_0^1\phi(y)\frac{y}{sy+1}dy=0\qquad(\Re(s)>0).$$

Expanding this last equation in the neighborhood of $s=0$ and equating to zero the powers of s, one obtains:

$$(1)\qquad \int_0^1\phi(y)\left\{y^2\log\left(1+\frac{1}{y}\right)-y\right\}dy=\tfrac{1}{2}a-\tfrac{1}{3}b,$$

$$(2)\qquad \int_0^1\phi(y)\left\{y^{n+1}\log\left(1+\frac{1}{y}\right)-y^n+\frac{y^{n-1}}{2}+\cdots+\frac{y}{n}\right\}dy=\frac{a}{n+1}-\frac{b}{n+2}\qquad(n=2,3,\cdots).$$

Similarly, multiplying (*) by s and letting $s\to\infty$ along the positive real axis, one obtains

$$(3)\qquad \int_0^1\phi(y)\left\{y\log\left(1+\frac{1}{y}\right)-2\right\}dy=a-\tfrac{1}{2}b.$$

The author shows that the approximate method of solution described by Menzel and Sen [same J. **110**, 1–11 (1949); these Rev. **11**, 185] can be recovered from equations (1)–(3) if we assume for $\phi(y)$ the form $\phi(y)=\sum_{r=3}^\infty a_r y^{r-3}$.

S. Chandrasekhar (Williams Bay, Wis.).

Code, A. D. Radiative equilibrium in an atmosphere in which pure scattering and pure absorption both play a role. Astrophys. J. **112**, 22–47 (1950).

In this paper the transfer of radiation in an atmosphere in which both electron scattering and continuous absorption play a role is considered. The polarization of the radiation field is properly allowed for. The equations of transfer are generalizations of those considered by the reviewer [same J. **103**, 351–370 (1946); these Rev. **7**, 494] and involve a parameter λ which is the ratio of the continuous absorption coefficient to the total absorption coefficient including electron scattering. The equations of transfer are solved in the second (and in some cases also the third) approximation in the method of solution in which the radiation field is divided into discrete beams. The state of polarization of the emergent radiation for various values of λ are numerically obtained and illustrated. Perturbation theories for $\lambda\to1$ and $\lambda\to0$ are also described. *S. Chandrasekhar.*

Malmfors, K. G. Unstable oscillations in an electron gas. Ark. Fys. **1**, 569–578 (1950).

The equations governing states of oscillation of electrons moving (on the average) in circular orbits with a linear velocity v_0 perpendicular to a magnetic field B under the influence of space charge fluctuations (in a direction x) are $d^2x/dt^2=m^{-1}f(x,t)-\omega dy/dt$, $dy/dt=\omega(x-x_0)$, $\partial f/\partial x=4\pi e^2n$, and the equation of continuity. In the foregoing equations n denotes the fluctuations in the number of electrons about a mean concentration n_0, $\omega=eB/m$ and the rest of the symbols have their usual meanings. Writing $x=x_0+\rho\cos(\varphi_0+\omega t)+\chi$ $(|\chi|\ll\rho)$, where ρ is the radius of curvature of the circular orbits so that $v_0=\rho\omega$ and χ is the small deviation from the undisturbed circular motion, the author shows that the equations of motion and continuity allow a solution of the form $f=ce^{2\pi i(x-zt)/a}$, where c, z, and a are constants, if the following characteristic equation for z/a is satisfied:

$$(*)\qquad \frac{1}{\lambda}\frac{d}{d\lambda}\left[\lambda\sum_{-\infty}^{+\infty}\frac{J_m(J_{m+1}-J_{m-1})}{m-2\pi z/a}\right]=2i\left(\frac{\omega}{\omega_0}\right)^2,$$

where $\lambda=2\pi\rho/a$ and $\omega_0=(4\pi e^2n_0/m)^{\frac{1}{2}}$ is the angular frequency of plasma oscillation in the absence of a magnetic field. In (*) the argument of the Bessel functions J_m is λ. From a consideration of this equation, the author shows that z/a is in general complex and that, therefore, the motion of electrons, all of which move with the same velocity perpendicular to a magnetic field, is unstable. *S. Chandrasekhar.*

Sauvenier-Goffin, E. Étude de la stabilité dynamique et de la stabilité vibrationnelle des naines blanches. Mém. Soc. Roy. Sci. Liège (4) **10**, 143 pp. (1950).

This monograph presents a careful and a detailed study of the dynamical and the vibrational stability of the degenerate stellar configurations ("white dwarfs"). In large part, this paper overlaps with the earlier investigations by the

author alone [Ann. Astrophysique 12, 39–51 (1949); these Rev. 11, 214] or in collaboration with Ledoux [Astrophys. J. 111, 611–624 (1950)].

Chapter I presents a general introduction to the subject and it is pointed out here that though the effective ratio of the specific heats tends to 4/3 in the extreme relativistic case, the configurations are nevertheless dynamically stable on account of the fact that the radius tends to zero while the mass tends to a finite limit (M_3).

In chapter II the virial theorem is first used in the manner of Ledoux and Pekeris [ibid. 94, 124–135 (1941)] to obtain an approximate expression for the period of infinitesimal adiabatic radial pulsation ($2\pi/\sigma$). Next the equation governing the pulsation is derived and the corresponding eigenvalue problem is formulated. While the results obtained here are the same as in the author's earlier investigation, the discussion of the behavior of the solution of the pulsation equation at its singular points is carried out very much more completely. A direct integration of the pulsation equation shows that the amplitude ξ_1 ($=\delta r_1/r_0$) of the fundamental mode of pulsation is remarkably constant through the configuration. This fact is later used in the investigation of stability.

In chapter III the vibrational stability is investigated. By using the orthogonality property of the solutions, ξ_k, belonging to the different eigenvalues, σ_k, the condition for the stability of the kth mode of pulsation is derived in the form

$$\int_0^R \frac{d}{dr}\left[\frac{x^2+2}{3(x^2+1)}(\delta\epsilon-\operatorname{div}\delta\mathbf{F})\right] r^2\xi_k dr<0,$$

where x^2 is proportional to the density and $\delta\epsilon$ and div $\delta\mathbf{F}$ are the variations in the rate of energy production, ϵ, and the flux of total energy $\mathbf{F}$ and the integral is extended over the whole configuration. The sign of the stability integral is investigated by considering, separately, the contributions from the degenerate central regions and the nondegenerate outer envelope. It is shown that if the energy is generated in the degenerate central regions according to a law of the form $\epsilon\propto\rho T^\nu$, then stability demands that $\nu\leqq 9.5$; similarly, if the energy is generated at the interface between the degenerate interior and the gaseous envelope, $\nu\leqq 2.6$.

S. Chandrasekhar (Williams Bay, Wis.).

Schlüter, Arnulf, und Biermann, Ludwig. Interstellare Magnetfelder. Z. Naturforschung 5a, 237–251 (1950).

The question of the spontaneous occurrence and amplification of magnetic fields by turbulence in interstellar space is considered. The paper is divided into two parts. In the first part, it is shown how the difference in the pressure gradients acting on ions (mass m_i) and electrons (mass m_e, charge $-e$) can give rise to a magnetic field. The order of the field created in this manner is estimated as follows: The current density $\mathbf{j}$ in the presence of a magnetic field $\mathbf{H}$ in a medium in which the velocity is $\mathbf{v}$ can be written as $\mathbf{j}=\sigma(\mathbf{E}^c+\mathbf{E}^i)$, where σ is the ordinary conductivity (in e.s.u.), $\mathbf{E}^c=\mathbf{E}+(\mathbf{v}\times\mathbf{H})/c$ is the electric field as measured by an observer moving with the velocity $\mathbf{v}$, and

$$(1)\qquad e\mathbf{E}^i=(m_e \operatorname{grad} p_e-m_i \operatorname{grad} p_i)/\rho+(m_i-m_e)d\mathbf{v}/dt,$$

where ρ is the density and p_i and p_e are the partial pressures of the ions and the electrons, respectively. Also, if $\mathfrak{D}\mathbf{H}/\mathfrak{D}t$ measures the rate of charge of the magnetic flux per unit area, the boundaries of which follow the motion represented by the velocity field $\mathbf{v}$, then

$$(2)\qquad \mathfrak{D}\mathbf{H}/\mathfrak{D}t=-c \operatorname{curl} \mathbf{E}^c=c \operatorname{curl} \mathbf{E}^i-c \operatorname{curl} (\mathbf{j}/\sigma).$$

According to equation (1), $e|\mathbf{E}^i|\cong m_H d|\mathbf{v}|/dt\cong m_H v^2/L$, where v is the mean turbulent velocity, L is the average size of the eddies, and m_H is the mass of the proton; from equation (2) it now follows that $|\mathfrak{D}\mathbf{H}/\mathfrak{D}t|\cong cm_H v^2/eL^2$. After the lapse of a time equal to the characteristic time of the turbulence, namely (L/v), we may expect a magnetic field of the order $H_0\cong cm_H v/eL$. With $v=5\times10^5$ cm/sec and $L=10^{21}$ cm, $H_0\cong10^{-19}$ gauss. In the second part of the paper the author considers the amplification of this magnetic field by the stretching of the magnetic lines of force by the fluctuating velocities due to turbulence. The author gives reasons for believing [cf. Batchelor, Proc. Roy. Soc. London. Ser. A. 201, 405–416 (1950); these Rev. 11, 699] that by this stretching mechanism, starting with a field H_0, the mean square field $\bar{H}^2$ at a later time T will be given by $\bar{H}^2=H_0^2e^{\alpha Tv/2L}$ ($\alpha\sim1$). For $T=3\times10^9$ years (which is the "age of the universe") and v and L having the same values as before, the amplification factor is of the order of 10^{12}–10^{15}. In other words, we may expect that at present the field strength in interstellar space is of the order of 10^{-7}–10^{-4} gauss. The importance of fields of this strength for various astrophysical and cosmic ray problems is discussed.

S. Chandrasekhar (Williams Bay, Wis.).

Boneff, N. Recherches nouvelles sur la distribution des formations sur la surface lunaire. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. (Math. Phys.) 44, 67–82 (1948). (French. Bulgarian summary)

RELATIVITY

Tonnelat, Marie-Antoinette. Théorie unitaire affine. I. Choix des tenseurs de base et obtention de l'équation fondamentale. C. R. Acad. Sci. Paris 231, 470–472 (1950).

The author points out that one set of the Einstein generalised field equations [The Meaning of Relativity, Princeton University Press, 1945; these Rev. 7, 87] may be written as

$$R^{\underset{+-}{\mu\nu}}{}_{;\rho}=\partial R^{\mu\nu}/\partial x_\rho+\Delta'^{\mu}_{\sigma\rho}R^{\sigma\nu}+\Delta'^{\nu}_{\rho\sigma}R^{\mu\sigma}-\Gamma_\rho' R^{\mu\nu}=0$$

without any additional conditions if the affine connection used by Einstein, $\Delta^{*\mu}_{\nu\rho}$, is replaced by another, $\Delta'^{\mu}_{\nu\rho}$, which differs from the first by an appropriately chosen tensor depending on the divergence of the antisymmetric part of $R^{\mu\nu}$.

A. H. Taub (Urbana, Ill.).

Tonnelat, Marie-Antoinette. Théorie unitaire affine. II. Résolution rigoureuse de l'équation fondamentale. C. R. Acad. Sci. Paris 231, 487–489 (1950).

The author solves the equations

$$r_{\underset{+-}{\mu\nu};\rho}=\partial r_{\mu\nu}/\partial x_\rho-\Delta'^{\sigma}_{\mu\rho}r_{\sigma\nu}-\Delta'^{\sigma}_{\rho\nu}r_{\mu\sigma}=0 \qquad (\mu,\nu,\rho=1,\cdots)$$

explicitly for the $\Delta'^{\sigma}_{\mu\nu}$ as functions of the $r_{\mu\nu}$. The method used is to decompose this set of 64 equations into two systems of linear equations, one of 24 equations for the antisymmetric parts of Δ' and one of 40 equations for the symmetric parts of Δ'. These systems are then solved explicitly. Straus [Rev. Modern Physics 21, 414–420 (1949); these Rev. 11, 216] has also solved this problem by another

method after remarking that the method used in this paper is feasible. *A. H. Taub* (Urbana, Ill.).

Tonnelat, Marie-Antoinette. Théorie unitaire affine. III. Les équations du champ. C. R. Acad. Sci. Paris **231**, 512–514 (1950).

The results of the two papers reviewed above are used to give explicit form to the field equations derived from the variational principle using the Lagrangean density $A=2(-\mathring{R})^{\frac{1}{2}}/\lambda$, where $\mathring{R}=\det\|R_{\mu\nu}(\mathring{\Delta})\|$ and $R_{\mu\nu}$ is the contracted Hermitian curvature tensor formed from the affine connection $\mathring{\Delta}$. It is assumed that the antisymmetric terms in the tensor $g_{\mu\nu}$ are such that their squares may be neglected.
A. H. Taub (Urbana, Ill.).

Ingraham, Richard L. Contributions to the Schrödinger non-symmetric affine unified field theory. Ann. of Math. (2) **52**, 743–752 (1950).

This paper deals with the nonsymmetric affine field theory of Schrödinger [Proc. Roy. Irish Acad. Sect. A. **51**, 163–171 (1947); 205–216 (1948); these Rev. **9**, 310]. The author puts the field equations into a form which, with the imposition of additional conditions, yields analogues of the electromagnetic and gravitational field equations. He discusses (without calculating) the equations of motion of particles. He also sets up an exact solution of the field equations. Unfortunately, the author does not discuss the question of how serious are the restrictions on the theory imposed by the additional conditions which he introduces. Furthermore, there is some doubt about the validity of the exact solution which he finds since it is based on the assumption of the existence of parallel vector fields in a space of (nonvanishing) constant curvature. *N. Rosen* (Chapel Hill, N. C.).

Möller, C. Sur la dynamique des systèmes ayant un moment angulaire interne. Ann. Inst. H. Poincaré **11**, 251–278 (1949).

Definitions of centre of mass for classical systems in restricted relativity are considered. For a system with no external forces but arbitrary internal interactions such definitions are possible [see the author, Communications Dublin Inst. Advanced Studies. Ser. A. no. **5** (1949); these Rev. **11**, 297] and throw light on certain equations of Mathisson [Acta Physica Polon. **6**, 163–200, 218–227 (1937)], and of Weyssenhoff [Weyssenhoff and Raabe, ibid. **11**, 7–18, 19–25 (1947); Weyssenhoff, ibid., 26–33, 34–45, 46–53 (1947)]. For a system subject to external forces other than those due to gravitation, the equations of motion are incompatible with the pair of conditions: (i) proportionality of momentum-energy vector and the velocity-of-centre-of-gravity vector, (ii) identity of centre of gravity with centre of mass for the reference system in which it is at rest. Thus it is impossible to give a satisfactory definition of centre of gravity. However, a representative point for the system may be defined. Equations of motion are derived which agree with those of Mathisson in their region of common validity. However, the representative point may move outside the system. For an electron the resulting uncertainty in the definition of the centre of gravity is of the order of the Compton wavelength. For a sufficiently small system subject to gravitational forces as the only external forces a nonambiguous meaning may be given to the idea of centre of gravity. *C. Strachan* (Aberdeen).

Bloch, Claude. Variation principle and conservation equations in non-local field theory. Danske Vid. Selsk. Mat.-Fys. Medd. **26**, no. 1, 30 pp. (1950).

A nonlocal field quantity is usually represented as an operator whose matrix elements $(x'|A|x'')$ are functions of two world points: x'^{μ} and x''^{μ}. These quantities may be regarded as functions $A(X,r)$, where $X^{\mu}=\frac{1}{2}(x'^{\mu}+x''^{\mu})$, $r^{\mu}=x'^{\mu}-x''^{\mu}$. The author associates with every nonlocal field quantity $A(X,r)$ a local density defined by $a(X)=\int A(X,r)dr$, and proposes to interpret the nonlocal field $A(X,r)$ by means of the associated $a(X)$. He shows that the field equations are derivable from a variation principle and that the conservation equations follow from the invariance of the Lagrangean function used. It is shown that two solutions of the field equations which differ only in the "internal rotation" terms lead to the same local densities. There is therefore no physical interpretation of these solutions by means of the local densities and the author proposes a Lorentz invariant supplementary equation for excluding these extra solutions. *A. H. Taub* (Urbana, Ill.).

Bergmann, Peter G., Penfield, Robert, Schiller, Ralph, and Zatzkis, Henry. The Hamiltonian of the general theory of relativity with electromagnetic field. Physical Rev. (2) **80**, 81–88 (1950).

The authors construct a Hamiltonian for field equations derivable from a Lagrangian which is a homogeneous quadratic function of the first derivatives of the field variables. The general method is applied to the case of the Einstein theory of gravitation with an electromagnetic field. This paper is dependent on the results of two previous papers [same Rev. (2) **75**, 680–685 (1949); Rev. Modern Physics **21**, 480–487 (1949); these Rev. **10**, 408; **11**, 299].
A. H. Taub (Urbana, Ill.).

Scherrer, W. Über den Einfluss des metrischen Feldes auf ein skalares Materiefeld. II. Helvetica Phys. Acta **23**, 547–555 (1950).

In a preceding paper [same Acta **22**, 537–551 (1949); these Rev. **11**, 467] the author discussed the centrally symmetric solution of the variation problem

$$\delta\int[(R-2\Lambda)\psi^2+4\omega g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}](-g)^{\frac{1}{2}}dx=0$$

for the case $\Lambda=0$, $\omega=0$. The discussion of the case $\Lambda=0$, $\omega\neq0$, here proceeds along lines very similar to the previous one. For all finite ω, the total energy is finite. For most values of the parameters entering the solution, the energy is distributed continuously, but in a particular limiting case is concentrated on a single world line with a Dirac-function-like singularity. The author remarks that, though the present theory, which introduces matter as a scalar density ψ^2, alleviates, it does not altogether resolve the deep-seated contradiction in Einstein's theory between the role of matter as a source of field and its role as a test body. He concludes, "We cannot avoid the conjecture that we have reached the end of the usefulness of the concept of field. We are confronted with the basic alternative: continuity or discontinuity? In particular, is it possible to consider the metric as a statistical phenomenon?" *A. J. Coleman.*

Clark, G. L. The external gravitational and electromagnetic fields of rotating bodies. Proc. Roy. Soc. London. Ser. A. **201**, 488–509 (1950).

The field equations $-8\pi T_{\mu\nu}=G_{\mu\nu}-\frac{1}{2}g_{\mu\nu}G$ may be regarded either (a) as partial differential equations for $g_{\mu\nu}$, given $T_{\mu\nu}$, or (b) as explicit formulae for the energy tensor $T_{\mu\nu}$, given $g_{\mu\nu}$.

In the present paper they are looked at in both ways. First, as (a), the author attempts to solve them with $T_{\mu\nu}=0$, using successive approximations with forms of $g_{\mu\nu}$ appropriate to the gravitational field of a rotating body [Proc. Cambridge Philos. Soc. **43**, 164–177 (1947); these Rev. **8**, 496]. The forms contain disposable constants, but these cannot be chosen so that the equations are satisfied. He then turns to plan (b), and examines the value of $T_{\mu\nu}$ given by the selected $g_{\mu\nu}$; this he is able to interpret (not uniquely) as the energy tensor of an electromagnetic field. He states: "The present investigation establishes Blackett's hypothesis that a rotating body produces an electromagnetic field." In view of the special character of the forms considered and his own uncertainty as to what rotation means in general relativity, the reviewer regards the word "establishes" as too strong.

J. L. Synge (College Park, Md.).

Clark, G. L. Note on the problem of a rotating mass of perfect fluid in relativity mechanics. Proc. Roy. Soc. London. Ser. A. **201**, 510–515 (1950).

The author considers a fluid with energy tensor of the usual form $T^{\mu\nu}=(\rho_{00}+p)(dx^\mu/ds)(dx^\nu/ds)-g^{\mu\nu}p$, and a motion given by $v_1=-\omega x_2$, $v_2=\omega x_1$, $v_3=0$, $v_4=1$ ($v_\mu=dx^\mu/dt$). Following plan (a) [see the preceding review] he cannot obtain an external field with $T_{\mu\nu}=0$. Turning to plan (b), he interprets the residual $T_{\mu\nu}$ as due to an electromagnetic field.

J. L. Synge (College Park, Md.).

Christov, Chr. Sur le problème du corps solide et les équations unitaires de l'électrodynamique et de la gravitation. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **43**, 43–112 (1947). (Bulgarian. French summary)

[Volume number misprinted **42** on title page. Author's name misspelled Chrisov in French summary.]

Newing, R. A. A six-vector development of some results in kinematical relativity. Quart. J. Math., Oxford Ser. (2) **1**, 153–160 (1950).

It is shown how some of the results in Milne's book "Kinematic Relativity" [Oxford, 1948; these Rev. **10**, 578] can be derived by using the 4-dimensional invariants of space-time, and, in particular, six-vectors.

A. E. Schild (Pittsburgh, Pa.).

★**Milne, E. A., and Dingle, H. Problèmes de philosophie des sciences. III. Théories nouvelles de relativité.** Actualités Sci. Ind., no. 1065. Hermann et Cie., Paris, 1949. 71 pp.

This is a verbatim report of lectures, and subsequent discussion, given at the Brussels Symposium in 1947. Milne gave an account of his kinematic theory of relativity [see Kinematic Relativity . . . , Oxford, 1948; these Rev. **10**, 578], and Dingle spoke on the function of time measurement in modern physics, with particular reference to his own thermal relativity theory. The discussion was mainly on kinematic relativity, the speakers being J. L. Destouches, G. Lemaître, H. Weyl, and others. *A. G. Walker*.

Gialanella, Lucio. L'espansione dell'universo controllo definitivo della teoria della relatività? Ricerca Sci. **20**, 911–924 (1950).

From a lecture given in March, 1950.

Finzi, Bruno. La nuova teoria relativistica unitaria di Einstein. Ricerca Sci. **20**, 901–910 (1950).

A summary of a lecture given at the Convegno Matematico di Milano in April, 1950. The full text will be published later.

Hartley, R. V. L. Matter, a mode of motion. Bell System Tech. J. **29**, 350–368 (1950).

Hartley, R. V. L. The reflection of diverging waves by a gyrostatic medium. Bell System Tech. J. **29**, 369–389 (1950).

The first paper contains a nonrigorous discussion of the behaviour of diverging spherical waves in a "gyrostatic ether." The equations describing the propagation of disturbances in this nonlinear medium are derived in an appendix. It is not clear that the approximations made in the discussion are consistent.

The second paper is a nonmathematical discussion of a nonlinear field representing a mechanical ether previously discussed by MacCullagh and Kelvin. Particles are represented by localised disturbances in this medium.

A. H. Taub (Urbana, Ill.).

MECHANICS

Carrizo Rueda, Jorge Eduardo. Kinematic study of the Cardan coupling. Ciencia y Técnica **115**, 74–88 (1950). (Spanish. French summary)

A graphical method for the determination, with any desired accuracy, of the ratio of the angular speeds of the driving and driven shafts in a Cardan coupling (a Hooke's joint, frequently called a universal joint) is given. The numerical values of the Fourier coefficients of the expansion of this functional relation in a series is determined by the graphical vectorial harmonic method of Ashworth-Harrison for selected values of the angles between the shafts.

M. Goldberg (Washington, D. C.).

Hain, K. Punktlagenzuordnungen mit gegebener Tangentenrichtung am Gelenkviereck. Ing.-Arch. **18**, 141–150 (1950).

It is known that a four-bar linkage can be designed so that a point on the moving connecting-rod can be made to pass through nine given points. The author has shown [Messtechnik **20**, 33–37, 59–61 (1944)] how to pass the curve through seven given points by the graphical constructions called "Punktlagenreduktionen." These methods can be specialized so that, in addition, the curve is tangent to given lines through some of the given points. Each tangent condition requires the removal of one point condition. However, there is frequent need for a simpler construction when the set of conditions are not as stringent. Therefore constructions are given for the following conditions: (a) up to five points and one tangent; (b) up to four points and two tangents. When freedom permits, other conditions may be imposed such as the fixing of a pivot point along certain lines. *M. Goldberg* (Washington, D. C.).

Artobolevskiĭ, I. I. Geometric methods for the solution of some problems of the theory of plane mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **1**, 129–152 (1947). (Russian)

As far as this reviewer can see, the chief result of this paper, whose obscurity is heavily enhanced by many puzzling misprints in diagrams and notations, is the following: The

hodograph of the connecting-rod plane BC of a four-bar linkage $ABCD$, if turned through $\frac{1}{2}\pi$ sgn ω, and centered at A, becomes homothetic with the plane BC, with center B and ratio ω_{BC}/ω_{AB} if $|V_B|=1$. The image α of the inflection circle is the featured contribution: it goes through A and the instant center P of BC, and its diameter Ak is the velocity of the instant center, k being the homologue of the inflection pole. Also the accelerations can be constructed in the hodograph plane. Fairly obvious constructions are given of the circle α for several linkage mechanisms, and of other elements. A last section is devoted to showing how accelerations can be constructed from the motion of the hodograph points. The paper proposes a device consisting of two sleeves pivoted to each other and sliding along two bars, in order to materialize their intersection point. In spite of the space devoted to its discussion, the results are entirely independent of this device. *A. W. Wundheiler* (Chicago, Ill.).

Čerkudinov, S. A. On the dead points of a driven member. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 2, 143–149 (1947). (Russian)

Two mechanism design problems are solved by the method of geometric loci of "centers", i.e., points equidistant from several positions of the same particle of a moving plane [cf. the author, same Trudy 1, 181–216 (1947); these Rev. 12, 136]. Problem 1: Given (a) two positions S_1, S_2 of a driving member S, (b) the corresponding positions of the driven member Q, and the instant center positions T_1, T_2 for S_1 and S_2; to find a connecting rod AB for S and Q so that Q_1, Q_2 are dead points for AB. Problem 2: Given (a) three positions S_1, S_2, S_3 of a driving member S, (b) the corresponding positions of the driven member Q, and (c) the instant center T_1 for S_1; to find a connecting rod AB for S and Q so that Q_1 is a dead point for AB. These problems were solved by H. Alt [Maschinenbau, der Betrieb 19, 173–176 (1940) = Getriebetechnik, Reuleaux-Mitteilungen 8, 17–20 (1940); Z. Angew. Math. Mech. 5, 337–346 (1925)] by a reduction to a four-position problem and the theory of Burmester curves. The present paper reduces them, respectively, to the loci of (1) intersections O of T_1A_1 and T_2A_2, with $OA_1=OA_2$, and arbitrary given T_1 and T_2, and (2) centers O for three corresponding particle positions A_1, A_2, A_3, with O collinear with T_1 and A_1, T_1 arbitrary given. Some simplified special cases are considered.

A. W. Wundheiler (Chicago, Ill.).

Čerkudinov, S. A. On the curvature of conjugate profiles of circular wheels. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 3, no. 9, 52–54 (1947). (Russian)

Let two conjugate profiles P_1, P_2 of two circular gears of centers O_1 and O_2 have the curvature centers M_1 and M_2 corresponding to the instant point of contact C. The four-hinge linkage $O_1M_1M_2O_2$ provides, to the second order, the same relative motion of O_1M_1 and O_2M_2 as that of the planes P_1, P_2. From this a simple proof is derived of the following theorem: The instant center of P_1 relative to P_2 is the foot of the perpendicular on M_1M_2 from $P=O_1M_1\times O_2M_2$ (Bobillier). *A. W. Wundheiler* (Chicago, Ill.).

Čerkudinov, S. A. The angle of transmission in four-hinge linkages. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 3, no. 9, 55–59 (1947). (Russian)

If O and C are the fixed hinges of a four-hinge linkage $OABC$, and OA is a crank, the angle $ABC=\mu$ is the "transmission angle." Grashof's "crank condition" is refined to express $\mu_0\leqq\mu\leqq 180°-\mu_0$ ($\mu_0<90°$) in terms of $r_1=OB/OC$, $r_2=CB/OC$, $r_3=AB/OC$. For a given r_1, the point (r_2, r_3) is confined to an area at the tip of an ellipse. This area reduces to a point when r_1 increases to $(1-\sin\mu_0)/\cos\mu_0$: This is the maximum of r_1 for a given μ_0. The case $\mu_0=0$ ($0<\mu<180°$ defines a single-crank linkage) is examined in some detail (the crank must be the smallest member). A (r_2, r_3)-graph for linkages of $\mu=30°$ and a set of r_1 values ($\leqq 0.58$) is given.

A. W. Wundheiler (Chicago, Ill.).

Čerkudinov, S. A. On some general questions of the synthesis of hinged mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 3, no. 10, 5–30 (1947). (Russian)

This paper attempts to generalize some concepts involved in linkage design. The dimensions (constants) of a linkage are identified as bar lengths, linear offsets (projections of distances between two hinges adjacent to the same slider on the normal to the slider), and angular offsets (angles between two adjacent sliders). The maximum number of nonvanishing dimensions is $3n-u-5$, where $n+1$ is the number of links and u the number of sliders. In the sequel, the discussion assumes an input (I) and output (O) link. Since the definitions given are not invariant under a transformation of the parameters defining the dimensions, an interpretation of the paper seems in order. For a standard input motion, the output need not depend essentially on all the dimensions. Transform the dimensions so that the output depends on some new parameters q essentially and only; the remaining ones must be chosen arbitrarily, and the author calls them "primarily assumable." The given output imposes certain relations between the q's. Transform the parameters q so that these relations involve the parameters s essentially and only. The remaining ones will be unrestricted by the design problem, and are called "secondarily assumable" (so will be all their functions). The s's are the "free dimensions" (actual design parameters). They may be joined by another variable specifying the associated "initial" positions of (I) and (O) (the author sees two new variables here). This ends the count of the unknowns involved in the design problem. The remaining half of the paper consists of highly obvious remarks on the number and kind of conditions that may be imposed on the input-output relationship in the attempt to approximate a given performance (higher order contacts, etc.).

A. W. Wundheiler (Chicago, Ill.).

Čerkudinov, S. A. Some applications of the method of inversion of motion. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 6, no. 24, 47–77 (1949). (Russian)

This paper deals with breeding of linkages generating nearly-circular coupler curves (or intermissions). If only the ratios of the link lengths are relevant, Roberts' theorem breeds three new linkages (with different link sequences) generating the same coupler curve. If a coupler curve (D) is a circular arc, the inverse motion has a circular coupler curve (D_0) of center D_0. If (D) is only nearly circular, so will be (D_0), the maximum deviation being the same for both curves, as well as the number of intersections with the true circle. Combined application of Roberts' transformation and inversion breeds 12 new linkages. If the mother linkage satisfies Grashof's condition of revolving, there will be three double-revolver linkages, six single-revolver linkages, and three double-rocker ones.

A formula for the crank swing angle corresponding to the nearly-circular portion is derived, and applied, in some detail, to one of the Chebyshev mechanisms. If $(D)=d$ is straight, inversion yields a motion in which one straight line d envelops a point D. The breeding theorems are extended to such mechanisms. Intermissions are obtained if the point D moving nearly in a circle of center D_0 is joined to a fixed point F by a two-bar element DEF with $|DE|=|DD_0|$. Then EF is subject to intermissions, and the breeding method can be applied to this mechanism. A brief study is made of the breeding from two slider-crank mechanisms with the block (1) sliding along a fixed base line and (2) turning about a fixed point of the base line.

A. W. Wundheiler (Chicago, Ill.).

Konvisarov, D. V. Analysis and new methods of construction of the trajectories of the points of a slider-crank mechanism. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **3**, no. 12, 5–22 (1947). (Russian)

A geometric study is made of the direct-acting slider-crank mechanism with an offset (crank OA, coupler AB; O fixed, B slides on a "horizontal" straight line not necessarily going through O). The two positions D, D' of a particle of the coupler plane corresponding to parallel positions of the coupler lie on a horizontal. The maximum ("principal") chord D_0D_0' is attained for a horizontal position of OA. The centers of the chords DD' lie on an elliptic arc, the "conjugate line" of (D). If D is on AB, denote by y its distance from the principal chord, and by S the length DD'. Then $S^2/4+CB^2y^2/AB^2=OA^2$. This, together with the equation of the conjugate line, leads to a simple pointwise construction of the trajectory of D. Three examples are given. Things are less simple if D is not on AB. A more involved pointwise construction of its trajectory is given and illustrated by means of a numerical-graphical example.

A. W. Wundheiler (Chicago, Ill.).

Levitskiĭ, N. I. Application of the least-square method to mechanism design. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **5**, no. 17, 40–68 (1948). (Russian)

The author advocates the least-square method for trajectory approximation by linkage curves. Although the maximum deviation is the quantity important for mechanisms, this method commends itself by the ease and scope of its application. Moreover, in sufficiently smooth cases the difference between average and maximum deviation will be practically negligible. The approximating curve $\psi(x)$ is cast in the form $\sum p_i\phi_i(x)$, where the ϕ_i do not depend on the design constants of the linkage. To approximate $f(x)$ the integral $\int[f(x)-\psi(x)]^2dx$ is minimized, which yields the familiar expressions for p_i in terms of $\int f\phi_k dx$ and $\int \phi_i\phi_k dx$. The method is profusely illustrated on the case of the Chebyshev "lambda" mechanism: four-hinge linkage $OABC$, OC fixed, $AB=BC$, the tracing point M lying on AB with $AB=BM$. The y coordinate of M is a linear combination of $[z(1-z)]^{\frac{1}{2}}$ and $[z^{-1}(1-z)]^{\frac{1}{2}}$, where $z=\sin^2\frac{1}{2}(ABC)$; further, $x^2+y^2=4(1-z)$. The coefficients depend on OA and OC. The curves of M are symmetric about the normal Cy through C to CO. The following trajectories are approximated: (1) two straight rays from a point on Cy; (2) a full ellipse (traced here without dead points); (3) a circular arc closed by a curve given graphically. All the algebraic details are given, as well as complete numerical illustrations.

A. W. Wundheiler (Chicago, Ill.).

Levitskiĭ, N. I. The synthesis of a hinged four-bar linkage with a given trajectory and three design parameters. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **6**, no. 23, 30–66 (1949). (Russian)

The method of the paper reviewed above is applied here to a problem sketched in an earlier note [Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1539–1542], and makes use of the formulas of another one [ibid. **1949**, 174–180]. Let $ABCD$ be the four-hinge linkage (AD fixed), CD is removed, a point M of BC is guided along the given curve, and C is required to move approximately (in the least-square sense) on a circle. The new length CD is expressed in terms of the coordinates of M and the constants of the mechanism, of which the old $CD=r$, AD, and the azimuth of AD are assumed to be subject to choice. The error $S=\sum_i(C_iD^2-r^2)$ is cast in the form $A\sum_i[F(x_i)-p_0\phi_0(x_i)-p_1\phi_1(x_i)-p_2]^2$, where only the p's depend on the three design constants in a known way, the ϕ's and F's being free of them. The p's are explicitly determined in terms of $F(x_i)$ and $\phi_k(x_i)$ by the least-square method. The formulas for the verification of the solution obtained are given. Two examples are computed in complete numerical detail: one for an open, another for a closed given trajectory. The same method is applied to the case when r, AD, and BC are the design parameters, illustrated again by a completely worked out numerical example.

A. W. Wundheiler (Chicago, Ill.).

Pinsker, I. Š. Approximate design of mechanisms with lower pairs. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **5**, no. 18, 34–83 (1948). (Russian)

A synthetic procedure of successive approximations is presented with all the trappings of modern mathematical rigor. Only a rough sketch of it, leaving out the precise assumptions, can be given here. The concept of curve distance is codified first. Each of two curve families, $\alpha(x, y; r_i)=0$ and $\beta(x, y; r_i)=0$, cover a domain of operations simply, as does a family (R) of straight lines (of azimuth ϕ). The distance is defined as the maximum length of the R segments between the curves in question, and is shown to be approximated by the (invariant) expression $\Phi=\alpha/A+\beta/B$, $A=\alpha_x\cos\phi+\alpha_y\sin\phi$, $B=\beta_x\cos\phi+\beta_y\sin\phi$, the error being bound by an expression in α, A,

$$\alpha_{xx}\cos^2\phi+2\alpha_{xy}\cos\phi\sin\phi+\alpha_{yy}\sin^2\phi,$$

etc. The quantity used as the measure of deviation is $[\Phi]=\max\Phi$ for a given set of r_i. If, for a choice of $r_i=r_{i0}$, $[\Phi_0]\geqq q$, where q is the acceptable error, put $F_1=\Phi_0+\sum(\partial\Phi/\partial r_i)_0p_i$ and determine p_i so that $[F_1]\leqq[\Phi]$ at $r_{i0}+p_i$. Then, as is shown, λ can be found so that $[\Phi_0(r_{i0}+\lambda p_i)]\leqq[\Phi(r_0)]$. This procedure is iterated until some $[\Phi(r_{in})]\leqq q$. The determination of the p_i may turn out to be involved enough to warrant replacing $[F_1]\leqq[\Phi_0]$ by one of these conditions: (a) F_1 is small; (b) F_1 vanishes at a number of points; (c) the least-square deviation of F_1 from zero is a minimum, for a set of points. The algebra involved is presented for the two last cases. The application of these procedures to linkages is generously illustrated, in complete numerical detail, for a number of four-bar linkage design problems.

A. W. Wundheiler (Chicago, Ill.).

Zinov'ev, V. A. Analytical methods for the determination of the positions of mechanisms of high classes. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **6**, no. 22, 61–74 (1949). (Russian)

The positions (azimuths) of the members of a multibar linkage are determined by equations of the type $\sum l_je^{i\phi_j}=0$,

where l_j and ϕ_j are the length and azimuth of the jth bar of a loop forming part of the mechanism. The author proposes to solve these equations by successive approximations, the corrections Δl_j, $\Delta\phi_j$ being determined by the equations $\sum \Delta l_j e^{i\phi_j} + \sum i l_j e^{i\phi_j} \Delta\phi_j = 0$. The errors are estimated and many examples are given. The convergence seems to be good. *A. W. Wundheiler* (Chicago, Ill.).

Hohenberg, Fritz. Über die Zusammensetzung zweier gleichförmigen Schraubungen. Monatsh. Math. **54**, 221–234 (1950).

The name "helicoidal curve" is chosen for the path traced by a point carried by a screw which rotates uniformly in a carriage which itself is moving uniformly along another screw of fixed axis. This combination of two uniform screw motions is expressed analytically in terms of the two angular velocities, the two pitch velocities, the angle between the axes and the distance between the axes. Special cases are conical, cylindrical and hyperboloidal helices and plane and spherical trochoidal curves. The instantaneous velocities and accelerations are derived analytically. The envelopes of carried lines and planes are considered. *M. Goldberg.*

Longhini, Pedro. On the statement and scope of the principle of virtual work. Ciencia y Tecnica **115**, 135–151 (1950). (Spanish. English summary)

This is a critical and expository article, the object being to state the principle of virtual work in its most general form, and to show how it applies to problems concerning the equilibria of dynamical systems of various kinds. In particular, the author discusses the applications of the principle to problems concerning elastic solids and concerning what he calls ideal dynamical systems. The latter are systems such that when the system undergoes a reversible virtual displacement, the virtual work done by the external constraints and the virtual work done by the internal forces both vanish. The discussion relating to elastic bodies includes derivations of some theorems concerning equilibrium, one of which is believed to be new. *L. A. MacColl.*

Gallissot, François. Sur l'origine du paradoxe de Painlevé dans les systèmes de points matériels ou de solides en mouvement avec frottement. C. R. Acad. Sci. Paris **230**, 2148–2150 (1950).

In a previous paper [same vol., 611–612 (1950); these Rev. **11**, 748] the Painlevé paradox was discussed from the point of view of the signature of a certain quadratic form. The same method is now reinterpreted in terms of the existence of at least one constraint equation, which, according to the main result of the paper, is necessary and sufficient for indeterminacies or impossibilities. *D. C. Lewis.*

Bradistilov, G. Existenz und Eigenschaften der periodischen Bewegungen des n-fachen Pendels in der Ebene. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **38**, 249–282 (1942). (Bulgarian. German summary)

Obrechkoff, N. Remarque sur les petits mouvements périodiques. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **38**, 293–296 (1942). (Bulgarian. French summary)

Grammel, R., und Zoller, K. Zur Mechanik der Peitsche und des Peitschenknalles. Z. Physik **127**, 11–15 (1950).

The problem of the crack of the whip is discussed on the basis of the theory of threads. Gravity and air resistance are neglected and certain additional simplifying assumptions are made. The authors show that then the speed v_1 of the knot at the end of the whip has as its maximum value $v_{1\,\max} = v([l\rho/m+1]^{1/2}-1)$, where m is the mass of the knot, ρ the mass per unit length, l the length of that part of the whip which can be considered as a thread without resistance against bending, and v the speed of its fixed end. The formula shows that a moderate value of v can cause a very large value of v_1, which explains the high supersonic velocities of the knot which have been observed. *P. Neményi.*

Hil'mi, G. F. Semidissipative motions in a system of n bodies attracting according to Newton's law. Doklady Akad. Nauk SSSR (N.S.) **71**, 1041–1044 (1950). (Russian)

A completely dissipative system, in the author's terminology, is a system in which the distance of each body from any other approaches infinity when the time increases beyond any limit. A system of $n-1$ material points subjected to mutual Newtonian attraction and possessing a total mass M is divided in two groups: (a) $P_0, P_1, \cdots, P_{k-1}$; (b) $P_k, P_{k+1}, \cdots, P_{n-1}$. Let m_i be the mass of P_i and r_{ij} the distance of P_i from P_j. Let further

$$M_i' = \sum_{j\neq i} m_i m_j, \quad M_i'' = \min\,\{m_i m_j/(m_i+m_j)\},$$

$$M_i = M_i'/M_i'', \qquad i = 0, 1, \cdots, k-1,$$

and $\rho_i(t) = \min_{j\neq i}\{r_{ij}\}$, $\sigma_i = \min_{j\neq i}\{dr_{ij}/dt\}$. The author proceeds to demonstrate the following theorem: If the constant energy integral H is positive and fulfills the following conditions: $r'_{ij}(0) - 4M_i/\sigma_i(0)\rho_i(0) > 0$, $i = 0, 1, \cdots, k-1$, $j\neq i$; $\sigma_i(0) > 0$, $\sigma_i^2(0) > 8M_i/\rho_i(0)$, $i = 0, 1, \cdots, k-1$; $\sum_{i=0}^{K-1}\sum_{j=0,\,j\neq i}^{n-1} m_i m_j\{r'_{ij}(0) - 4M_i/\sigma_i(0)\rho_i(0)\} - {}^{2}2MH = \eta > 0$; then $\rho_i(t) \to \infty$ when $t\to\infty$ for all $i = 0, 1, \cdots, k-1$, and the group composed of $P_k, \cdots, P_{n-1}$ cannot be a completely dissipative subsystem. If the number of bodies in the latter group is only two, it can be shown that they form a stable subsystem. *L. Jacchia* (Cambridge, Mass.).

Hydrodynamics, Aerodynamics, Acoustics

✱Kuethe, A. M., and Schetzer, J. D. Foundations of Aerodynamics. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. ix+374 pp. $5.75.

This text is designed to introduce the student to the basic concepts of fluid mechanics which underlie the science of aerodynamics. The emphasis of the treatment is upon physical concepts rather than upon mathematical theory and the prerequisite mathematical knowledge is quite moderate, the book being on about the senior level. Except for a final chapter on the aerodynamic characteristics of wings the subject matter is sharply distinct from that of a course in technical aerodynamics; thus, the book is not intended to serve as an aerodynamics text proper. The first six chapters of the book deal with perfect fluids through thin airfoil theory and Prandtl wing theory. Five chapters are spent on the basic elements of compressible flows and five on introduction to viscous fluids and turbulence. One chapter on viscous compressible fluids and the chapter on wing characteristics complete the text.

Criticisms of the book are few. The use of vector analysis is somewhat anomalous, as many important formulas are expressed in vector form, while the student is expected to develop but a cursory ability in this field. The general

three-dimensional statement of the Prandtl-Glauert rule is avoided. The adiabatic assumption is unnecessarily made in the derivation of Bernoullis' equation and Helmholtz' circulation law is used for compressible fluids, having been established only for incompressible fluids. The controversial principle that the Tollmien-Schlichting instability is responsible for turbulence is perhaps expressed without sufficient reservations. On the whole the book is admirably suited to the purpose. Particular good points include a good statement of the important Euler momentum theorem, an excellent discussion of the factors affecting transition from laminar to turbulent flow, a discussion of shock-boundary layer interaction, an appendix on dimensional analysis, adequate inclusion of representative and basic references, and a good index. Its aim, to give the aerodynamics student conceptual understanding to supplement technical knowledge, is the text's best point. *W. D. Hayes.*

Birkhoff, G., Plesset, M., and Simmons, N. Wall effects in cavity flow. I. Quart. Appl. Math. 8, 151–168 (1950).

The authors calculate wall corrections to the drag coefficient for channel flows past polygonal obstacles with trailing infinite cavities. Similar results in this direction are contained in the work of Réthy [Math. Ann. 46, 249–272 (1893)], Valcovici [Thesis, Göttingen, 1913], and von Mises [Z. Verein. Deutsch. Ingenieure 61, 447–452, 469–474, 493–498 (1917)], but the authors consider that their own formulas provide a simpler basis for computation and that their numerical results are more exact and more extensive. The methods, based on the usual Schwarz-Christoffel mappings, apply to all flows with free streamlines whose hodographs are circular sectors, but the specific numerical results center on flows past a flat plate in a closed channel, a free jet, and in a semi-infinite closed channel terminating in a free jet. A number of computations and graphs show among other things that the wall effects, when based on comparison of the drag coefficient in the channel with its usual value in an infinite fluid, are greater for the closed channel than for a free jet of the same width, and that they are considerably reduced (of course, only formally) by basing the drag coefficient on the downstream velocity. *D. Gilbarg.*

Moriya, Tomijiro. On Blasius formulae referred to moving axes. J. Jap. Soc. Appl. Mech. 3, 81–85 (1950).

The case treated is the two-dimensional one of an infinite cylinder of arbitrary section moving through a body of incompressible perfect fluid with constant translational and angular velocities normal to and about a line parallel to the cylinder axis. The force and moment are easily expressed in integrals taken over the contour. The author then assumes that a conformal transformation, in series form, maps the region exterior to the cylinder on that exterior to a circle, and proceeds to determine the complex potential. These methods are similar to those used by R. M. Morris in a more general problem [Philos. Mag. (7) 23, 757–762 (1937); 24, 445–453 (1937); Proc. Roy. Soc. London. Ser. A. 188, 439–463 (1947); these Rev. 8, 542 and references given there]. In some special cases, such as motion of flat plates, the integrated formulas for lift and moment reduce to known results. The author proposes to treat centrifugal pumps by this method in another paper. *W. R. Sears.*

Arakawa, H. The vorticity equations in the spherical and cylindrical coordinates. Geophys. Mag. 16, 1–4 (1948).

The author writes down the formula for the material derivative of the vorticity vector in spherical and in cylindrical coordinates, taking into account the earth's rotation and neglecting the effect of viscosity. *C. Truesdell.*

Funaioli, E. Sul progetto di schiere alari di caratteristiche prefissate. Aerotecnica 30, 114–119 (1950).

A method is presented and illustrated for the generation of a family of profile elements of a cascade of profiles in two dimensions with certain prescribed characteristics in incompressible flow. The characteristics of the family of cascades are prescribed in terms of parameters in the plane of the canonical flow about a circle and are related to parameters of the flow in the plane of the cascade. *C. Saltzer.*

Mangler, W. General solution of Prandtl's boundary-layer equation. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1278, 20 pp. (1950).

[Translated from Lilienthal-Gesellschaft für Luftfahrtforschung, Bericht 141, 1941.] Prandtl's boundary layer equations with arbitrary pressure distribution are transformed into the form:

$$\frac{\partial^3\zeta}{\partial\eta^3}-\frac{\partial^2\zeta}{\partial\xi\partial\eta}=\frac{\partial^2\zeta}{\partial\eta^2}\frac{\partial}{\partial\xi}\left(\frac{\zeta}{\bar{a}}\right)-\frac{\partial\zeta}{\partial\eta}\left[\frac{\partial^2}{\partial\xi\partial\eta}\left(\frac{\zeta}{\bar{a}}\right)+\frac{1}{2\bar{a}}\frac{d\bar{a}}{d\xi}\frac{\partial}{\partial\eta}\left(\frac{\zeta}{\bar{a}}\right)\right],$$

where

$$\zeta=(y/x-\psi/Ux)(Ux/U_\infty L)\bar{a}\sqrt{Re},\quad \xi=\int_0^x(U(x)/U_\infty L)dx,$$

$$\eta=(yU(x)/LU_\infty)\sqrt{Re},\quad \bar{a}=(U/U_\infty)^2,\quad Re=LU_\infty/\nu,$$

x, y being the coordinate axes parallel and perpendicular, respectively, to the surface of the body, ψ the stream function, $U(x)$ the velocity of the main flow, U_∞ the velocity of the free stream, L a characteristic length, and ν the coefficient of kinematic viscosity. A numerical method for solving this equation is developed, based on a known solution of the equation of heat conduction. The technique is applied to the boundary-layer flow over a circular cylinder. The iteration has been carried to the third approximation with an accuracy that insures its usefulness for the general problem. *Y. H. Kuo* (Ithaca, N. Y.).

Illingworth, C. R. Steady flow in the laminar boundary layer of a gas. Proc. Roy. Soc. London. Ser. A. 199, 533–558 (1949).

The author makes a systematic investigation of the laminar boundary layer of a gas, including (in certain parts) the effect of gravity. The transformation of von Mises is used as the basic method. In the case with Prandtl number equal to unity, the viscosity coefficient proportional to the temperature, and the boundary insulated, it is shown that the boundary layer problem for a gas may be made identical, whatever be the main stream, with a suitable problem for an incompressible fluid. In other cases, convenient methods of solution are given and carried out. The problems of free convection at a flat plate and laminar flow in plane jets and wakes are also treated. *C. C. Lin* (Cambridge, Mass.).

Illingworth, C. R. Unsteady laminar flow of gas near an infinite flat plate. Proc. Cambridge Philos. Soc. 46, 603–613 (1950).

This paper investigates the unsteady laminar compressible flow near an infinite flat plate. It is found that the von Mises transformation is still valid in such boundary-layer equations. If gravity is not considered, the boundary layer equations with the usual Prandtl approximation hold both for the flat plate and for the rotating circular cylinder. In

the case of an infinite constant-temperature plate moving at constant velocity, solutions are obtained for an arbitrary Prandtl number. Along the same line of approach, the diffusion of a vortex sheet is also analyzed. A flat plate with variable velocity is also considered with temperature-dependent viscosity under two conditions of the plate: (a) constant-temperature; and (b) heat-insulated. Also, a plate starting from rest with constant acceleration is analyzed, including the effect of gravity. The relative importance of a free convection due to gravity and forced convection due to viscosity is discussed. A solution is obtained for the case of a free convection current set up near a plate which is at rest in a mass of gas of entirely different temperature. *C. C. Chang* (Baltimore, Md.).

Cooke, J. C. The boundary layer of a class of infinite yawed cylinders. Proc. Cambridge Philos. Soc. **46**, 645–648 (1950).

Following Prandtl [Ministry of Aircraft Production, Reports and Translations no. 64 (1946)] and others, the author calculates the three-dimensional laminar boundary-layer flow around yawed infinite cylinders. The class of cylinders considered has the potential-flow speed $U=cx^m$ and thus includes pointed sections and symmetrical wedges. The "spanwise" component $v(x, z)$ is calculated by numerical quadratures from values tabulated for unyawed boundary-layer flows, and is tabulated for several values of β $[=2m/(m+1)]$. For $\beta=1$, the results agree with those of Prandtl [loc. cit.] and of Sears [J. Aeronaut. Sci. **15**, 49–52 (1948); these Rev. **9**, 476], who considered $U=ax+bx^3$.
W. R. Sears (Ithaca, N. Y.).

Kito, Fumiki. On secondary vortex generated in a bent-pipe of elliptical cross-section. J. Jap. Soc. Appl. Mech. **3**, 73–75 (1950). (Japanese. English summary)

We consider a laminar flow of a viscous liquid through a bent-pipe whose cross-section has elliptical form. Assuming that the radius R of bend is large in comparison with the radii (a, b) of cross-section, the author has obtained the amount of secondary vortex flow which will take place in the cross-section of the pipe. *Author's summary.*

Booth, F. The cataphoresis of spherical particles in strong fields. J. Chem. Phys. **18**, 1361–1364 (1950).

D. C. Henry [Proc. Roy. Soc. London. Ser. A. **133**, 106–129 (1931)] considered the motion of a charged conducting solid sphere suspended in an electrolyte, produced by an external electric field. The hydrodynamic problem was solved under the assumptions: (a) that the applied field and the field due to the double layer formed between the solid and the liquid are superposable; (b) that the motion is slow so that the inertia force is negligible; (c) that the thickness of the double layer is small compared with the radius of curvature of the body. The equilibrium condition between frictional and electric forces determines the cataphoretic velocity of the sphere. This same problem is now extended to the case of larger Reynolds number by Oseen's approximation; the correction to Henry's result was found generally small for small dimensions and weak fields.
Y. H. Kuo (Ithaca, N. Y.).

Hausenblas, H. Die nichtisotherme laminare Strömung einer zähen Flüssigkeit durch enge Spalte und Kapillarröhren. Ing.-Arch. **18**, 151–166 (1950).

In the laminar flow of a fluid the viscous energy dissipation will cause a rise in fluid temperature unless heat is removed at a proper rate. Assuming that such heat removal occurs so that the temperature remains constant in the direction of flow, the author obtains analytically the temperature and velocity profiles for flow in a slit and a tube assuming the reciprocal of the viscosity to vary linearly with temperature. The results are obtained in terms of Bessel functions and are expressed on a dimensionless basis by use of a reference constant viscosity. The author erroneously identifies this reference state as isothermal flow.
N. A. Hall (Minneapolis, Minn.).

Christianovich, S. A., and Yuriev, I. M. Subsonic gas flow past a wing profile. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1250**, 29 pp. (1950).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Meh.] **11**, 105–118 (1947); these Rev. **9**, 543.

Sears, W. R. Transonic potential flow of a compressible fluid. J. Appl. Phys. **21**, 771–778 (1950).

This is a general review of the state of transonic flow theory. The significance of the continuous irrotational flows which can be evaluated is considered from the point of view of (a) influence of the boundary layer, (b) nonexistence of neighbouring solutions, (c) temporal instability. Item (c) contains the principal contributions which are new, including an unpublished theory of Y. H. Kuo. But according to a letter in a later issue [same vol., 1340 (1950)], this theory has now been drastically revised, and its conclusions agree with those inferred from (b) [see, e.g., A. Busemann, J. Aeronaut. Sci. **16**, 337–344, 434 (1949); these Rev. **11**, 223].
M. J. Lighthill (Manchester).

Perl, W., and Klein, Milton M. Theoretical investigation and application of transonic similarity law for two-dimensional flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2191**, 42 pp. (1950).

The similarity rules for the two-dimensional steady potential flow in the transonic speed range have been derived by von Kármán. In the sense of small perturbations, one simplified assumption has been made that the boundary at the body may be satisfied on the body axis. To show the validity of the similarity rules even without such an assumption, some iteration procedures of small parameter have been adopted. In each step of the successive approximation the boundary condition is satisfied exactly at the body. The resulting solution is of the same form required by the existing transonic similarity theory. Thus, the similarity rules are proved true even without the simplified assumption on the boundary condition. Furthermore, the transonic similarity form has been reduced to the exact numerical results obtained by Kaplan for the cases of a circular arc and the so-called Kaplan section. *C. C. Chang.*

Whitham, G. B. The behaviour of supersonic flow past a body of revolution, far from the axis. Proc. Roy. Soc. London. Ser. A. **201**, 89–109 (1950).

The linearized theory of axisymmetric flow assumes straight parallel characteristics. It thus gives an incorrect picture of the flow at large distances from the axis, where the characteristics in fact are curved, and diverge. The author develops a solution asymptotically valid for large distances from the axis, which establishes the form of the characteristics of the shapes of the front and rear shocks, and the approximate nature of the flow between the shocks. The method is based on the substitution in the exact flow equations of expansions suggested by the linearized theory.
D. P. Ling (Murray Hill, N. J.).

Wang, Chi-Teh, and Chou, Pei-Chi. Application of Biezeno-Koch method to compressible fluid flow problems. J. Aeronaut. Sci. 17, 599–600 (1950).

The equation of continuity for cylindrically symmetrical irrotational isentropic flow is solved approximately. The method used is the following: The velocity potential φ is expanded in terms of N functions satisfying the boundary conditions and the expansion involves N arbitrary parameters. The parameters are chosen so that the residuals obtained by substituting φ into the equation of continuity when integrated over various regions of space are a minimum.

A. H. Taub (Urbana, Ill.).

Imai, Isao. Tables useful for the numerical calculations of the air stream at high speeds. J. Phys. Soc. Japan 3, 342–345 (1948).

The following quantities are tabulated for isentropic flow: Mach number M; $|1-M^2|^{\frac{1}{2}}$; density ratio ρ/ρ_0; pressure ratio; and $q\rho/\rho_0$, all as functions of q, the ratio of the gas speed to the speed of sound at $M=1$. Values of q are taken from 0 to 2.44 by steps of 0.1; this covers a range of M from 0 to 25.3 ....

W. R. Sears (Ithaca, N. Y.).

Martin, John C. The calculation of downwash behind wings of arbitrary plan form at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2135, 47 pp. (1950).

This is a useful account of procedures for calculating downwash behind a wing in supersonic flow. The work is presented from first principles. Of special value will be the explicit expressions for the downwash distributions due to lifting lines of general and special shapes. Simplifying approximations, beyond the fundamental linearisation of the equations, are given for certain formulae, and these are critically examined by comparison with the more exact expressions.

M. J. Lighthill (Manchester).

Czarnecki, K. R., and Mueller, James N. An approximate method of calculating pressures in the tip region of a rectangular wing of circular-arc section at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2211, 24 pp. (1950).

Walker, Harold J., and Ballantyne, Mary B. Pressure distribution and damping in steady pitch at supersonic Mach numbers of flat swept-back wings having all edges subsonic. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2197, 62 pp. (1950).

von Mises, R. On the thickness of a steady shock wave. J. Aeronaut. Sci. 17, 551–554 (1950).

The problem of computing the conditions within a steady shock wave by means of the one-dimensional Navier-Stokes equations has been treated by several authors. The present paper, however, gives a more elegant treatment of the problem. By using the nondimensional temperature t and square of the velocity v as variables, the mathematics of the problem is reduced to simple terms. It is then shown that the shock wave is represented by an integral curve in the (v, t)-plane joining the singular points of the differential equation in that plane. This explains how it is possible to specify both the initial point and the end point of the solution in spite of the fact that the differential equation is a first order equation. Based upon these considerations, the author gives simple and narrow limits for the thickness of the shock without any elaborate integration.

H. S. Tsien (Pasadena, Calif.).

Cabannes, Henri. Calcul de la courbure au sommet de l'onde de choc attachée dans un écoulement de révolution. C. R. Acad. Sci. Paris 231, 325–326 (1950).

Consider axisymmetric supersonic flow past a body of revolution having an attached bow shock. If the body has a conical portion at the nose, the bow shock also has a conical portion and there is a region of conical flow, terminated downstream by the characteristic drawn from the first point of curvature of the body I and intersecting the shock wave in A. The author undertakes to determine the curvature of the shock wave at A in terms of the curvature of the body at I. Results are given, but not the details of the analysis.

W. R. Sears (Ithaca, N. Y.).

Brier, Glenn W. The statistical theory of turbulence and the problem of diffusion in the atmosphere. J. Meteorol. 7, 283–290 (1950).

The author demonstrates the difficulty of arriving at a satisfactory theory of diffusion involving correlation functions which are easily measurable. He also describes a recent experiment where measurements were made on the scatter of a cluster of balloons and points out some of the statistical concepts that should be considered in the analysis and interpretation of such data.

C. C. Lin.

Sweet, P. A. The effect of turbulence on a magnetic field. Monthly Not. Roy. Astr. Soc. 110, 69–83 (1950).

In this paper the effect of turbulence on the magnetic field acting on an incompressible conducting fluid is considered. First, it is shown that if $\mathbf{K}$ is a solenoidal vector and $\mathbf{u}$ denotes the velocity of the stream lines (defined by $d\mathbf{r} \propto \mathbf{K}$) normal to itself, then $\partial\mathbf{K}/\partial t = \operatorname{curl}(\mathbf{u}\times\mathbf{K})$. If $\mathbf{w}=\mathbf{v}-\mathbf{u}$, where $\mathbf{v}$ denotes the material velocity, then the electromagnetic equation

$$\text{(1)} \qquad \operatorname{curl}\mathbf{H} = 4\pi\sigma[v\times\mathbf{H}+c\mathbf{E}^{\text{ext}}-c\operatorname{grad}V-\partial\mathbf{A}/\partial t]$$

can be rewritten in the form

$$\text{(2)} \qquad \operatorname{curl}\mathbf{H} = 4\pi\sigma[\mathbf{w}\times\mathbf{H}+c\mathbf{E}^{\text{ext}}-c\operatorname{grad}V],$$

where $\mathbf{E}^{\text{ext}}$ denotes the impressed E.M.F., $\mathbf{A}$, the vector potential, and V is an arbitrary function of position and time. Similarly, the equation of motion takes the form

$$\rho D\mathbf{v}/Dt = -\operatorname{grad}p+\rho\mathbf{g}-\sigma H^2\mathbf{w}_n+\sigma c\mathbf{E}^{\text{ext}}\times\mathbf{H},$$

where $\mathbf{w}_n$ denotes the component of $\mathbf{w}$ normal to $\mathbf{H}$ and the other symbols have their usual meanings. The author next considers the effect of a magnetic field $\mathbf{H}$ symmetrical about the z-axis on turbulence which is assumed to distort the lines of force about their mean position $\bar{u}=\lambda=$ a constant, where $u(\bar{\omega}, z, t)$ represents the distorted line of force ($\bar{\omega}$ denotes the normal distance from the z-axis). By a suitable averaging of equation (1), the author shows that the equation governing the mean field $\mathbf{H}_m$ becomes

$$\text{(3)} \quad \operatorname{curl}\mathbf{H}_m+\frac{\sigma_T}{\rho\bar{\omega}^2}\operatorname{grad}\left(\frac{\rho\bar{\omega}^2}{\sigma_T}\right)\times\mathbf{H}_m \cong 4\pi\sigma_T[\mathbf{v}\times\mathbf{H}_m+c\mathbf{E}^{\text{ext}}-\partial\mathbf{A}_m/\partial t],$$

where $\sigma_T=\sigma/q\gamma^2$, $\gamma=\int_{\bar{u}=\lambda}ds/\int_{u=\lambda}ds$, $q=\bar{H}\overline{(1/H)}$ $(\cong 1)$, and $\bar{H}$ is the mean value of the scalar strength of the distorted field. The physical meaning of γ is that it represents the ratio of the length of a distorted line of force to the corresponding length of the mean line; it is therefore also a measure of the constriction of the tubes of force consequent on the distortion. In deriving equation (3), the assumption has been made that in the case considered turbulence can be pictured

as occurring in tight folds in surfaces of revolution about the z-axis. Since in a highly turbulent region $\gamma \gg 1$, the author concludes from the similarity of equation (1) and (3) that the effect of turbulence is to reduce the effective conductivity by a very large factor. This result is illustrated by examples and its importance for astrophysical problems is stressed. *S. Chandrasekhar* (Williams Bay, Wis.).

Dungey, J. W. A note on magnetic fields in conducting materials. Proc. Cambridge Philos. Soc. **46**, 651–654 (1950).

The equations governing the electromagnetic field (neglecting the displacement current) in an infinitely good conductor are $\mathbf{E} = -\mathbf{v} \times \mathbf{H}/c$ and $\partial \mathbf{H}/\partial t = -c \operatorname{curl} \mathbf{E}$. Using these equations, the author shows that the integral $\int_S \mathbf{H} \cdot d\mathbf{S}$ over a surface S bounded by a closed convex curve σ is constant as we follow σ with the motion, provided the velocity in the fluid is everywhere continuous and single-valued. The proof is similar to the proof of Thomson's theorem on vorticity in hydrodynamics.
S. Chandrasekhar (Williams Bay, Wis.).

Gutman, L. N. On thermal disturbances in horizontal air flow. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 277–286 (1950). (Russian)

A line source of heat located in a horizontal air flow gives rise to a pattern of vertical free convection superimposed on the forced convective wake from the heat sources. This pattern is investigated first for laminar flow with the free convective distortion entering as a convection. The velocity and temperature distribution is obtained as a power series in a parameter proportional to the thermal expansion coefficient of the gas. Several terms in this series are obtained explicitly. The same problem is considered for turbulent flow assuming a Prandtl-Taylor mixing length proportional to the horizontal component of distance from the heat source. *N. A. Hall* (Minneapolis, Minn.).

Stommel, Henry. An example of thermal convection. Trans. Amer. Geophys. Union **31**, 553–554 (1950).

The oceanographic problem of two-dimensional free convective currents with a surface condition of sinusoidal longitudinal temperature variation is briefly stated. An approach to an infinite series solution in powers of the coefficient of thermal expansion of the fluid is outlined.
N. A. Hall (Minneapolis, Minn.).

Chester, W. The propagation of a sound pulse in the presence of a semi-infinite open-ended channel. I. Philos. Trans. Roy. Soc. London. Ser. A. **242**, 527–556 (1950).

The author's summary is as follows. "The behaviour of a sound pulse approaching and progressing beyond the open end of a semi-infinite channel is discussed. A discussion of diffracted waves is created at the open end for which a general formula is obtained, by operational methods, when the pulse originates inside the channel. With the aid of a simple reciprocity relation the asymptotic behaviour of these diffracted waves can be used to deduce the form of the wave returning along the channel when the original pulse approaches the open end from an arbitrary direction. Ultimately the returning wave becomes sensibly plane and separates into regions of length equal to the width of the channel, the form of the potential depending on the number of diffracted waves which contribute to each particular region. Explicit expressions are obtained for the potential in the first two regions at the head of the returning wave and for the third region when the pulse originates inside the channel. The case of an initial velocity distribution given by the Heaviside unit pulse is treated in detail."
A. E. Heins (Pittsburgh, Pa.).

Chester, W. The propagation of a sound pulse in the presence of a semi-infinite, open-ended channel. II. Proc. Roy. Soc. London. Ser. A. **203**, 33–42 (1950).

This is a continuation of some work by the author [see the preceding review]. In part I, a study of the disturbance produced inside a semi-infinite parallel plate medium when a transient sound wave approached the open end was discussed. In the present paper, the author discusses the asymptotic behaviour of the disturbance at large distances from the wave front. For an incident Heaviside unit pulse, the wave inside also tends to behave like a unit pulse.
A. E. Heins (Pittsburgh, Pa.).

Elasticity, Plasticity

Oldroyd, J. G. Finite strains in an anisotropic elastic continuum. Proc. Roy. Soc. London. Ser. A. **202**, 345–358 (1950).

The author continues the researches initiated in a previous paper [same Proc. Ser. A. **200**, 523–541 (1950); these Rev. **11**, 703]. He first derives general stress-strain relations for an elastic body [the unwary reader should be warned that the deceivingly simple result (eq. (9)) is virtually impossible to use, since it is valid only when the initial ("convected") and final ("fixed") coordinates are made to coincide]. For the isotropic case the author shows that his results reduce to those of Murnaghan [Amer. J. Math. **59**, 235–260 (1937)] and Rivlin [Philos. Trans. Roy. Soc. London. Ser. A. **240**, 459–490, 491–508, 509–525 (1948); **241**, 379–397 (1948); these Rev. **10**, 168, 340], and thus that the stress-strain relations of these two authors are effectively the same. [Reviewer's note. The general stress-strain relations were obtained by Kirchhoff [Akad. Wiss. Wien, S.-B. Mat.-Nat. Cl. **9**, 762–773 (1852)], Kelvin [Philos. Trans. Roy. Soc. London. Ser. A. **153**, 583–616 (1863) = Papers, v. 3, Clay, London, 1890, pp. 351–394], and others, and were well known in the nineteenth century; the explicit form for isotropic media, rediscovered by Rivlin [loc. cit.], was first given by Finger [Akad. Wiss. Wien S.-B. IIa **103**, 1073–1100 (1894)].]

The author next makes the statement that "the discussion by previous writers, concerning the finite straining of a continuum . . . does not appear to include a derivation of compatibility conditions . . . ," after which he uses the customary method of the Riemann tensor to obtain them. [Reviewer's note. The literature of this subject is extensive: the earliest reference is O. Manville, Mém. Soc. Sci. Phys. Nat. Bordeaux (6) **2**, 83–162 (1904) = Thèse, Bordeaux, 1903; cf. A. Signorini, Ann. Mat. Pura Appl. (4) **22**, 33–143 (1943), chapter 1, § 20; these Rev. **8**, 240.]

The author states that the usual definition of incompressibility is not compatible with his conception of perfect elasticity. He introduces a distortion tensor [similar to that used by Mooney, J. Appl. Phys. **19**, 434–444 (1948)] in terms of which he defines an "almost incompressible material," approaching incompressibility by a limiting process. Finally he studies the torsion of an anisotropic cylinder of

a material possessing a strain energy which is linear in the distortion tensor and its inverse, obtaining the result that the assumed deformation is possible if and only if the twelve assumed elastic coefficients reduce to five in a certain way. [Reviewer's note. This anomalous result may be explained by the fact that the stress-strain relations used by the author for the anisotropic material are valid only for isotropic materials, as is clear from the discussion of Murnaghan [loc. cit.] and Signorini [Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 56–71 (1942), see pp. 66–67; these Rev. **8**, 421].] *C. Truesdell* (Bloomington, Ind.).

Kubo, Ryogo. Large elastic deformation of rubber. J. Phys. Soc. Japan **3**, 312–317 (1948).

Guided by the structure theory of high polymers [H. James and E. Guth, Physical Rev. (2) **59**, 111 (1941), and many later papers], the author proposes to regard rubber as an incompressible perfectly elastic material with a strain energy proportional to the sum of the squares of the three principal extensions. [This material was simultaneously introduced by Rivlin, Philos. Trans. Roy. Soc. London. Ser. A. **240**, 459–490 (1948); these Rev. **10**, 168, who called it the "neo-Hookean material"; the author's paper was read in 1945, and Rivlin's results date from 1946 or earlier.] Introducing an arbitrary hydrostatic pressure as a Lagrangean multiplier corresponding to the constraint of incompressibility, the author obtains statical and dynamical equations. After noting agreement with the results of the statistical theory for simple extension and for pure shear, he applies his equations to the following special cases: (A) spherical shell; (B) cylindrical shell; (C) propagation of transverse waves. [The author's result for case (A) has subsequently been greatly generalized by Green and Shield, Proc. Roy. Soc. London. Ser. A. **202**, 407–419 (1950); these Rev. **12**, 218 and for case (B) by Rivlin, Philos. Trans. Roy. Soc. London. Ser. A. **242**, 173–195 (1949); these Rev. **11**, 627.] *C. Truesdell* (Bloomington, Ind.).

Reissner, Eric. On a variational theorem in elasticity. J. Math. Physics **29**, 90–95 (1950).

Zur Bestimmung der Komponenten des Spannungstensors $\sigma_x, \tau_{xy}, \cdots$, und der Verschiebungen u, v, w wird hier ein Variationsproblem von der Form

$$\delta\left\{\iiint\limits_V\left[\sigma_x\frac{\partial u}{\partial x}+\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\cdots-W(\sigma_x,\cdots)\right]dV - \iint\limits_{S_1}[\bar{p}_xu+\bar{p}_yv+\bar{p}_zw]dS\right\}=0$$

aufgestellt. Dabei bedeuten W die Formänderungsenergie, ausgedrückt durch die Spannungsgrössen, $\bar{p}_x$, $\bar{p}_y$, $\bar{p}_z$ die Komponenten der Oberflächenkräfte und S_1 jenen Bereich der Oberfläche, wo äussere Kräfte wirken. Dieses Variationsproblem wird verwendet zur Herleitung der Differentialgleichung für die durchgebogene Platte. Dabei wird für die Spannungen ein bereits früher vom Verfasser verwendeter Ansatz gemacht [Quart. Appl. Math. **5**, 55–68 (1947); diese Rev. **8**, 547], wo die einzelnen Spannungskomponenten als lineare, bzw. quadratische Funktionen des Abstandes z von der mittleren Plattenebene angesetzt werden. Durch Integration nach z ergeben sich Formeln für die, in bestimmter Weise gemittelten Verschiebungen. Es ergibt sich eine Vereinfachung gegenüber der Herleitung in der oben zitierten Arbeit. *P. Funk* (Wien).

Kondo, Kazuo. The geometry of the perfect tension field. II. J. Jap. Soc. Appl. Mech. **3**, 85–88, 96 (1950).

[For part I see J. Soc. Appl. Mech. Japan **2**, 3–4 (1949); these Rev. **11**, 282.] The author continues his researches on stress systems in which two of the three principal stress vectors vanish. Identifying such a state of stress approximately with that arising in the free deformation of an elastic sheet, he obtains a new formulation of the principle of minimum energy for this special case. *C. Truesdell.*

Kondo, Kazuo. On the dislocation, the group of holonomy and the theory of yielding. J. Jap. Soc. Appl. Mech. **3**, 107–110 (1950).

This paper purports to connect the phenomenon of yielding with the mapping of a Riemannian space onto a tangent flat space. The reviewer is unable to follow the author's remarks in detail or to find a statement of any specific conclusions. *C. Truesdell* (Bloomington, Ind.).

Zerna, W. Allgemeine Grundgleichungen der Elastizitätstheorie. Ing.-Arch. **18**, 211–220 (1950).

The major part of this paper contains an exposition of some of the elements of tensor analysis for the engineering reader. One section is a brief exposition of the theory of elasticity, for the most part restricted to the infinitesimal case. *C. Truesdell* (Bloomington, Ind.).

Tzénoff, I. Sur la déformation d'un élément infiniment petit d'un système matériel continu. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **43**, 373–394 (1947). (Bulgarian. French summary)

***Lur'e, A. I. Statika tonkostennyh uprugih oboloček. [Statics of Thin Elastic Shells].** OGIZ, Moscow-Leningrad, 1947. 252 pp.

Chapter 1. The equations of equilibrium of an elastic symmetrically loaded shell of revolution. Chapter 2. Solution of the fundamental differential equations for a shell of the simplest geometrical form. Chapter 3. Approximate solution of the fundamental differential equations of a symmetrically loaded shell of revolution. Chapter 4. Arbitrarily loaded cylindrical shell. *Table of contents.*

Mitrinovitch, Dragoslav S. Mise en correspondance d'un problème non résolu de la théorie de l'élasticité avec un problème résolu par Darboux et Drach. C. R. Acad. Sci. Paris **231**, 327–328 (1950).

Neményi and the reviewer [Proc. Nat. Acad. Sci. U. S. A. **29**, 159–162 (1943); Truesdell, Trans. Amer. Math. Soc. **58**, 96–166 (1945), § 19; these Rev. **5**, 84; **7**, 231] reduced the general equilibrium problem of the classical membrane theory of thin elastic shells of revolution to that of solution of the single differential equation $f''/f=A^2p$, where A is an arbitrary constant and p is a given function determined by the form of the shell. The author shows that this problem in turn is equivalent to that of solving $\eta''/\eta=\phi+A^2$, where ϕ is another given function. While the reviewer's inverse method of solution yielded simple exact solutions for numerous infinite families of shell forms, it was haphazard. The author states that the results concerning $\eta''/\eta=\phi+A^2$ previously obtained by Darboux [Leçons sur la théorie générale des surfaces . . . , v. 2, 2d ed., Gauthier-Villars, Paris, 1915, p. 210], Drach [same C. R. **168**, 47–50, 337–340 (1919)], and the author [Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. **6**, 121–156 (1939)] enable one systematically to construct infinite sequences of shell forms for which the equilibrium problem can be solved by quadratures. *C. Truesdell.*

Ilieff, Lübomir. Über das Gleichgewicht von elliptischen Membranen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 39, 409–426 (1943). (Bulgarian. German summary)

"In der vorliegenden Arbeit wird das folgende Problem gelöst: Gegeben ist eine elliptische ebene homogene Membran, die im Ruhezustand in der horizontalen (x, y)-Ebene gelegen ist und die eine Einzellast im Mittelpunkt trägt. Gesucht wird ihre Gleichgewichtslage, falls sie am Rande eingeklemmt ist." *From the author's summary.*

Bloh, V. I. Flexure of an unbounded elastic plate with doubly periodic loading. Doklady Akad. Nauk SSSR (N.S.) 73, 45–47 (1950). (Russian)

Consider a thin unbounded elastic plate subjected to a normal load which is doubly periodic in a nonorthogonal system of rectilinear coordinates. A solution is given for the displacement of the middle surface in the form of an absolutely and uniformly converging series. *H. I. Ansoff.*

Gradwell, C. F. Asymmetrical bending of tapered disks. The solution of a problem in disk bending occurring in connexion with gas turbines. Aircraft Engrg. 22, 209–212 (1950).

Explicit solutions are obtained for a problem of transverse bending of elastic, circular-ring plates of variables thickness. It is assumed that the inner edge of the plate is built-in and twisted about a diameter and that the outer edge remains plane and is (1) built-in, (2) simply supported. The thickness of the plate is assumed to vary like some power of the radial coordinate r. *E. Reissner* (Cambridge, Mass.).

Abramyan, B. L. Torsion and flexure of prismatic bars with a hollow rectangular cross-section. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 265–276 (1950). (Russian)

Consider a homogeneous elastic prismatic bar with a hollow rectangular cross section. Auxiliary stress functions are introduced which permit a solution of the Poisson equation by separation of variables. The stress function is represented by series expansions and the coefficients are determined from an infinite system of linear algebraic equations. This system is shown to be regular for rectangular bars with a rectangular hole even when the hole is reduced to a slit. Numerical bounds on the torsional rigidity and the stress in a square bar are compared to engineering values for holes of varying size. The problem of flexure of a rectangular hollow bar is next solved by the same method.

H. I. Ansoff (Santa Monica, Calif.).

Salvati, Michele. Sul comportamento dei tubi sottili all' azione simultanea della pressione idraulica e di forze esterne. Atti Relaz. Accad. Pugliese Sci. N.S. 5, 175–198 (1947).

The author gives elementary approximate solutions of the following problem: An infinitely long thin-walled pipe is under the simultaneous influence of a hydrostatic pressure distribution and of a force system causing a change of shape of the mean fiber, such as two opposite vertical edge loads or two opposite vertical loads uniformly distributed in the horizontal projection. He investigates the circumstances under which the application of the principle of superposition does not lead to excessive errors. *P. Neményi.*

Danilovskaya, V. I. Thermal stresses in an elastic half-plane arising from a sudden heating of its boundary. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 316–318 (1950). (Russian)

Consider an unstressed elastic half-space at a uniform temperature in which the temperature of the free edge is suddenly increased to a new constant value. The paper under review considers the problem of the subsequent temperature distribution, taking into account the initial transient caused by inertia of the elastic medium (if the inertia term is neglected, the problem is clearly trivial). The one-dimensional equations of motion and heat flow are solved by use of Laplace transforms. The stress at a fixed point is shown to decrease from zero until the elastic wave originated at the edge at $t=0$ arrives there. At that time the stress undergoes a jump to a positive value and thereafter decays to zero. *H. I. Ansoff.*

Mindlin, Raymond D., and Cheng, David H. Nuclei of strain in the semi-infinite solid. J. Appl. Phys. 21, 926–930 (1950).

Mindlin's solution for the single force in the interior of a semi-infinite solid is differentiated, integrated, and results superposed to obtain stress functions for 40 nuclei of strain. The corresponding stress states have zero surface traction on the plane boundary. Results are expressed in terms of the Galerkin vector for the following cases: (A) single force; (B) double force; (C) double force with moment; (D) line of double forces with moment; (E) center of dilatation; (F) line of centers of dilatation; (G) doublet; (H) linearly varying line of doublets with strength proportional to distance from the origin; (I) center of rotation.

D. C. Drucker (Providence, R. I.).

Mindlin, Raymond D., and Cheng, David H. Thermoelastic stress in the semi-infinite solid. J. Appl. Phys. 21, 931–933 (1950).

The solution for a center of dilatation [see the preceding review] in the semi-infinite solid is applied to the case of an expanding spherical inclusion in an elastic half-space. Also, it is shown that for the semi-infinite body with stress-free surface the potential problem is the same as for the infinite body. The plane boundary is freed of stress by applying a center of dilatation, a double force, and a doublet, each of proper strength, at the image point of each center of dilatation in the body. *D. C. Drucker* (Providence, R. I.).

Sen, Bibhutibhusan. Note on the stresses produced by nuclei of thermo-elastic strain in a semi-infinite elastic solid. Quart. Appl. Math. 8, 365–369 (1951).

The solution is derived for the center of dilatation in an isotropic, elastic body bounded by a traction-free plane. The result could have been obtained, by a simple process of differentiation and addition, from the solution for the single force [cf. the reviewer, J. Appl. Phys. 7, 195–202 (1936)]. The author relates the result to Goodier's theory of thermoelasticity and notes that, for a distribution of centers of dilatation, the total effect may be found by integration. He does not observe that despite the singularities at image points, which are required to maintain the surface traction-free, the integration to be performed is identical with that for the infinite body [see the preceding review].

R. D. Mindlin (New York, N. Y.).

O'Rourke, R. C., and Saenz, A. W. Quenching stresses in transparent isotropic media and the photoelastic method. Quart. Appl. Math. 8, 303–311 (1950).

The integrated relative phase retardation of light rays passed through equatorial sections of symmetrically stressed cylinders or spheres is shown to be related to the components of stress in the interior through an Abel integral equation. Solution of the equation yields simple quadrature formulas for the interior stresses. *R. D. Mindlin.*

Gol'denblat, I. I. Some new problems in the dynamics of structures. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1950, 819–833 (1950). (Russian)

This paper presents a survey of the results of research in the dynamics of structures obtained at the Central Institute for Scientific Research on Industrial Structures. This paper is organized into three parts: (1) quasi-harmonic oscillations; (2) oscillations of elastic systems under the action of moving loads; and (3) nonlinear oscillations. A series of tests is discussed confirming the existence of quasi-harmonic resonance predicted by the Mathieu equations. The motion of two trains at equal speeds across a bridge and the motion of a liquid inside of an elastic pipe are next taken as examples of the theoretical investigations into the action of moving loads. Itis shown that, if the inertia of the moving mass is taken into account, the structure will become unstable for sufficiently high velocities. The coupling between the vertical and the horizontal oscillations of a suspension bridge are next used to illustrate an important case of nonlinear oscillations. Conditions under which a transfer of energy between the modes takes place have been experimentally verified at the institute. A bibliography of the quoted results is appended. *H. I. Ansoff* (Santa Monica, Calif.).

Sakadi, Zyuro, and Takizawa, Eiiti. Mathematical treatment on the decay of a vibrating system due to the emission of sound wave into the surrounding medium. J. Phys. Soc. Japan 3, 235–241 (1948).

This paper deals with the mathematical problem of a vibrating system immersed in a fluid. The special cases considered are: radial vibration of an elastic sphere in the atmosphere; infinitely extended membrane (the media on the two sides may be different); torsional vibration of an infinite circular cylinder in a viscous fluid; transverse vibration of an infinite, stretched string; infinitely extended plate; extensional vibration of an infinite thin rod in a viscous fluid; radial vibration of a circular cylinder in the atmosphere. *Y. H. Kuo* (Ithaca, N. Y.).

★Hill, R. The Mathematical Theory of Plasticity. Oxford, at the Clarendon Press, 1950. ix+356 pp.

Time, temperature, Bauschinger, hysteresis, and size effects are all explicitly ruled out and the major but not exclusive emphasis is on ideal plasticity. However, a wide range of topics is discussed in this treatise. Mathematical proofs, experimental evidence, and the practical evaluation of theory are kept in balance from the study of stress-strain relations, variational principles, and the questions of uniqueness, to the solving of practical metal-forming problems. A few relatively simple elastic-plastic solutions are included for prismatic bars, and for thick cylindrical tubes and spherical shells. However, most of the chapters deal with plastic-rigid techniques and solutions developed by E. H. Lee and the author. An extensive discussion is given of the slip line fields for plane problems. The necessity for complete solutions is stated strongly and repeated warning is given against the error of thinking in terms of static determinacy only and not considering velocity conditions as well. A number of solutions are given in detail for which the configuration remains geometrically similar as the plastic deformation proceeds. Among the miscellaneous subjects covered are: machining, hardness tests, notched bars, normal and oblique necking, earing, and anisotropy. Tensor notation is used throughout. As is proper for a book on the mathematical theory of plasticity, the physics of metals is covered only by reference to treatises on the subject. A very brief appendix on suffix notation, the summation convention, and hyperbolic differential equations is stated to be sufficient for the reader who is familiar with the elementary theory of elasticity.

D. C. Drucker (Providence, R. I.).

Hill, R. A theory of the plastic bulging of a metal diaphragm by lateral pressure. Philos. Mag. (7) 41, 1133–1142 (1950).

In the reviewer's opinion, this paper offers the first satisfactory theoretical treatment of the problem indicated in the title. The mathematical technique used by the author is essentially a perturbation procedure with the ratio of the thickness to the radius of the plate as parameter. The resulting formulas apply to a perfectly plastic material as well as to materials with linear strain hardening and seem to agree well with experimental data. *W. Prager.*

Kuznecov, D. Concerning the lines of development of the theory of plasticity. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1950, 760–769 (1950). (Russian)

The first part of this paper presents a reply by the author to the criticism directed by Ilyushin at the "physics" school of plasticity [same Izvestia. Otd. Tehn. Nauk 1949, 1753–1773; these Rev. 11, 484]. If the present author's quotations from Ilyushin's paper are to be credited, the speciousness of the charges and countercharges must disqualify this from being a scientific argument. The second part of the paper presents an attempt to define and compare the points of view of the school of crystal plasticity and of the mathematical school. Here again the author's prejudice seems to lead to an oversimplification of the conceptual approach used in the mathematical school. Thus a mathematical theory is regarded as a curve-fitting problem concerned with extensions of the empirical results obtained for simple states of stress to more complex distributions. *H. I. Ansoff.*

Gubkin, S. I. Some basic problems of the science of plasticity. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1950, 770–784 (1 plate) (1950). (Russian)

This paper presents critical comments on Ilyushin's article referred to in the preceding review. Since, unlike Kuznecov [see the preceding review], the author is not replying to personal criticisms, the present paper contains some thoughtful comments on the development of the science of plasticity. Ilyushin is criticized for failing to give recognition to the contributions of physics, physical chemistry, and metallurgy to the studies of plastic behavior of matter. Some fundamental characteristics of the phenomenon of plasticity are stated and on the basis of these a comprehensive research program is proposed for the joint development of a science of plasticity by physicists, physical chemists, and applied mathematicians.

H. I. Ansoff (Santa Monica, Calif.).

Il'yušin, A. A. Remarks on some papers devoted to a critique of the theory of plasticity. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1950**, 940–951 (1950). (Russian)

The remarks concern criticisms of an earlier paper by the author [same journal **1949**, 1753–1773 (1949); these Rev. **11**, 484], by Kuznecov [see the 2d preceding review], Gubkin [see the preceding review], Ratner [same journal **1950**, 435–450], and Kiškin [ibid., 266–278 (1950)].

Handelman, G. H. Torsion of thin-walled closed cylinders beyond the elastic limit. J. Aeronaut. Sci. **17**, 499–507, 518 (1950).

The stresses throughout the cylinder are assumed to have passed the elastic limit, so that the entire cylinder is in the plastic state. The stress-strain law used in the plastic range is of the deformation type and only loading conditions are admitted. Under these assumptions the author uses the usual engineering approximations of elastic torsion of thin-walled tubes to derive formulas for the distribution of the shear stress, the shear strain, and the warping function. The problem is then reduced to an equivalent elastic tube through the introduction of an artificial thickness. The results are extended to include multicell sections and numerical examples are given. The limiting assumptions under which the results are valid are scattered throughout the text. *H. I. Ansoff* (Santa Monica, Calif.).

Kochendörfer, Albert, und Seeger, Alfred. Theorie der Versetzungen in eindimensionalen Atomreihen. I. Periodisch angeordnete Versetzungen. Z. Physik **127**, 533–550 (1950).

Zur theoretischen Untersuchung der Kristallplastizität wird eine lineare Atomreihe betrachtet in der "Versetzungen" vorhanden sind, d.h. dass innerhalb einer Länge L_0, in der im ungestörten Falle N Atome sein müssten, dort $N-1$, bzw. $N+1$, vorhanden sind. Die Wirkung des ungestörten Gitters auf diese Atomreihe wird als periodisches Potential berücksichtigt; ausserdem üben die Atome der gedehnten (oder zusammengedrückten) Kette aufeinander elastische Kräfte aus. Für die potentielle Energie folgt daher

$$E_p=\sum_n\{\Phi^0[1-\cos 2\pi\alpha^{-1}(z_n^0+q_n)-(1-\cos 2\pi\alpha^{-1}z_n^0)] \\ +\tfrac{1}{2}c(z^0_{n+1}+q_{n+1}-z_n^0-q_n-a)^2-\tfrac{1}{2}c(z^0_{n+1}-z_n^0-a)^2\},$$

wo die z_n^0 die Anfangslagen, die q_n die Verschiebungen bedeuten, α ist die Periode des Potentials, und a der (ungestörte) Atomabstand. Es muss nicht $\alpha=a$ sein. Da sich q_n nur langsam mit n ändert, kann man die obige Summe in ein Integral umformen, ebenso kann man auch die hier auftretende kinetische Energie schreiben. Aus diesen Energieausdrücken folgt mit Hilfe des Hamiltonschen Prinzips die Euler-Lagrangesche Differentialgleichung

$$c\partial^2q/\partial n^2-m\partial^2q/\partial t^2=2\pi\Phi^0\alpha^{-1}\sin 2\pi\alpha^{-1}(z^0+q),$$

wo m die Masse eines Atoms ist. Mit Hilfe von neuen Variabeln folgt

$$\partial^2u/\partial x^2-\partial^2u/\partial y^2=\sin(Z^0+u).$$

Durch entsprechende weitere Änderung von u kann man noch die Konstante Z^0 eliminieren und ausserdem mit Hilfe der Schreibweise $w=c_1x+c_2y$ die zeitabhängige Lösung auf die statische zurückführen. Also genügt es statt der erhaltenen hyperbolischen Differentialgleichung die gewöhnliche $d^2u_{a0}/dx^2=\sin u_{a0}$ zu lösen, die nach der ersten Integration die Form $(du^*_{a0}/dx)^2=k^{-2}(1-k^2\sin u^*_{a0})$ annimmt $(u^*_{a0}=\frac{1}{2}(u_{a0}-\pi))$ und die sich durch elliptische Integrale erster Gattung lösen lässt. Der physikalische Sinn der Lösungen und besonders das Problem der endlichen Atomreihen wird besprochen. *T. Neugebauer* (Budapest).

Leibfried, Günther. Über den Einfluss thermisch angeregter Schallwellen auf die plastische Deformation. Z. Physik **127**, 344–356 (1950).

Der Wirkungsquerschnitt σ einer Versetzung in einem Kristallgitter wird von der Grössenordnung des Quadrates der Gitterkonstante angenommen und die daran auftretende Streuung der thermischen Wellen wird berechnet. Unter alleiniger Berücksichtigung einer Transversalwelle $\tau=\tau_0\cos(\kappa_s x-\omega t)$, wo τ die Schubspannung, κ_s die Wellenzahl, und ω die Kreisfrequenz bedeuten, folgt, dass das Versetzungszentrum kleine durch die Gleichung $\xi=\xi_0\cos(\omega t-\alpha)$ beschriebene Schwingungen ausführt. Durch Berechnung der Leistung der Welle an der Versetzung, einerseits mit Hilfe von τ und $\bar{\xi}$, anderseits aus der Energiedichte und Ausbreitungsgeschwindigkeit c_t der Schallwellen und gleichsetzen der Resultate, folgt ein Zusammenhang zwischen ξ_0, α, und τ_0. Aus der Schubspannung und des erhaltenen Zusammenhanges folgt dann weiter für die mittlere Kraft am Orte der Versetzung $\bar{K}=\kappa_s\sigma c_t\tau_0^2/2\omega G$, wo G den Schubmodul bedeutet. Da ausserdem die Versetzung mit der Geschwindigkeit v vor der Schallwelle wegläuft, so folgt $\bar{K}(v)=\bar{K}(0)-\kappa_s\sigma v\tau_0^2/2\omega G$, wo nur das zweite Glied interessant ist und wofür man nach der im letzten Abschnitt durchgeführten Mittelung über alle thermischen Wellen $\bar{K}=-\sigma v\bar{\epsilon}/10c_t$ erhält. Hier bedeutet $\bar{\epsilon}$ die mittlere Energiedichte und aus der Abschätzung der Grössenordnung derselben und der bekannten elastischen Grenze (aus der man $\bar{K}$ erhält) folgt, dass $v\sim 0.1c_t$ ist, was bezüglich der Theorie der plastischen Deformation ein wichtiges Ergebnis ist. Weiter folgt aus dem berechneten Werte des quadratischen Mittelwertes der Schubspannung, also aus $\overline{\tau^2}=\frac{1}{2}Gz\lambda^{-3}kT$ (z Anzahl der Atome je Elementarzelle) und gleichsetzen am Schmelzpunkt von $(\overline{\tau^2})^{\frac{1}{2}}$ und der aus der theoretischen Schubfestigkeit folgenden Spannung, eine Formel für die Schmelztemperatur (T_s), $G\lambda^3/kT_s=95$, die bis auf einen Zahlenfaktor mit der Lindemannschen Formel übereinstimmt. *T. Neugebauer* (Budapest).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Slevogt, H. Die Verzeichnung als Funktion der Bildlage. Optik **6**, 321–326 (1950).

The author gives formulas for the change of distortion with object distance and specializes for the third order aberration theory. *M. Herzberger* (Rochester, N. Y.).

Drodofsky, M., und Slevogt, H. Zur Theorie des anallaktischen Punktes. Optik **7**, 23–26 (1950).

The authors discuss the anallactic point of a telescope, that is, the point from which to calculate the distance of a distant object of given size seen under a small angle such that this distance obeys the equation $a=f_0/\beta$, where β is the magnification with which the distant object is imaged. This point, which coincides with the object side focal point,

usually does not remain constant if the telescope is focused. The author suggests methods to have this point as stable as possible if the focusing of the whole system is done by the shifting of the inner lens of the optical system.

M. Herzberger (Rochester, N. Y.).

Weber, P.-E. Einfallwinkel (ϵ) als Funktion des Ablenkungswinkels (δ) und der Brechungszahlen (n, n'). Optik 7, 169–171 (1950).

The author gives an elementary formula which permits the calculation of the incident angles of a ray necessary to give a desired deviation. *M. Herzberger.*

Iijima, Taizo. Theory of inter-reflection between two infinite parallel planes. Jap. Sci. Rev. Ser. I. 1, 9–14 (1949).

The author investigates the illumination generated by a number of light sources between two parallel infinite planes with perfect diffusing reflection under the assumption that the light sources neither absorb light nor disturb the propagation (an assumption which is essentially justified as long as the light sources are small with respect to the distances between the two planes). The mathematical formulation of the problem leads to the problem of solving a system of integral equations. *M. Herzberger* (Rochester, N. Y.).

Scandone, F. Théorie de la transmission et de la réflexion dans les systèmes de couches minces multiples. J. Phys. Radium (8) 11, 337–341 (1950).

The author gives an elementary and concise theory of the transmission and reflection of multiple layers of film. Obtaining first the general theory for normal incidence with the help of a matrix product, the author generalizes his theory for an arbitrary angle of incidence, obtaining as a special case the well known formula of Fresnel. He investigates the case of absorbing media in which the refractive index has to be taken as a complex number and discusses, for normal incidence, layers with continuous refractive index.

M. Herzberger (Rochester, N. Y.).

Nijboer, B. R. A. The diffraction theory of optical aberrations. II. Diffraction pattern in the presence of small aberrations. Physica 13, 605–620 (1947).

The author develops his theory of diffraction for an off-axis point in rotationally symmetric optical systems under the assumption that the aberrations are small. The mathematical treatment of the problem with the help of so-called circle polynomials leads to a clear insight into the difficult problem. The circle polynomials introduced by the author are orthogonal for the interior of a unit circle and are very closely related to the Legendre polynomials. They were introduced into optical theory by Zernike, and the author shows that the development of the aberrations according to these functions leads, for small aberrations, to a method permitting the balancing of errors of different orders.

M. Herzberger (Rochester, N. Y.).

Glaser, W., und Grümm, H. Die Kaustikfläche der Elektronenlinsen. Optik 7, 96–120 (1950).

The authors give the formula of the caustic corresponding to an off-axis point in an electron optical system. Restricting themselves to the third order aberrations, they find, as would be expected from Hamilton's theory, that the forms of the caustic surface are equivalent to those discussed by Finsterwalder for geometrical optics, if one adds to them the rotation of the caustic around the axis produced by the anisotropy of the magnetic field. A numerical calculation is made for a magnetic lens. A model is constructed and photographed. *M. Herzberger* (Rochester, N. Y.).

Toraldo di Francia, G. Parageometrical optics. J. Opt. Soc. Amer. 40, 600–602 (1950).

The author suggests a theory intermediate between geometrical optics and diffraction theory based upon the assumption that the square of the wavelengths of light, but not the wavelengths themselves, can be neglected. As an example of his parageometrical optics the author discusses the meaning of Fermat's principle, the theorem of Malus-Dupin, Liouville's theorem, and the theory of a Soret grating. *M. Herzberger* (Rochester, N. Y.).

Fournet, G., et Guinier, A. L'état actuel de la théorie de la diffusion des rayons X aux petits angles. J. Phys. Radium (8) 11, 516–520 (1950).

Luneberg, Rudolf K. Asymptotic evaluation of diffraction integrals. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-15.** ii+52 pp. (1949).

In a previous report [same series, Rep. No. **EM-14** (1949); these Rev. 11, 630] the author has developed a theory for expressing any steady-state electromagnetic field in the form of an asymptotic series in which the basic variable is the wavelength. The present report serves first as an illustration of how the theory on asymptotic expansions is applicable to physical problems. Secondly, it gives a wave formulation of a somewhat generalized problem of diffraction through lenses. The leading terms in the asymptotic expansions representing the behaviour of the field either for very small wavelength at finite distances from the lens or for arbitrary wavelength at infinite distances from the lens are precisely the terms given by geometrical optics. Diffraction effects, corresponding to high-order terms of the asymptotic expansions, are not explicitly evaluated. The author refers to an analogous evaluation of diffraction integrals by van Kampen [Physica 14, 575–589 (1948)].

C. J. Bouwkamp (Eindhoven).

Blanc-Lapierre, André, et Perrot, Marcel. Diffraction et quantité d'information. C. R. Acad. Sci. Paris 231, 539–541 (1950).

Communication theory is applied to the correspondence between an object and its diffraction image through a small opening. The "quantity of information" is defined, and is evaluated for circular and square openings, for which it is inversely proportional to the square of the linear resolving power. *J. L. Doob* (Urbana, Ill.).

Müller, Claus. Zur mathematischen Theorie elektromagnetischer Schwingungen. Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1945/46**, no. 3, 56 pp. (1950).

The author examines the Maxwell equations for homogeneous and nonhomogeneous media. In particular, there is a study of the following topics: (a) Huygens' principle; (b) reradiation conditions; (c) integral equation representation of the field; and (d) uniqueness theorems.

A. E. Heins (Pittsburgh, Pa.).

Keller, Joseph B. Reflection and transmission of electromagnetic waves by thin curved shells. J. Appl. Phys. 21, 896–901 (1950).

The problem of the scattering of an arbitrary electromagnetic field by a given obstacle is reduced to the solution

of a pair of inhomogeneous linear integro-differential equations. A formal procedure for obtaining a power-series solution of these equations is given for the case of a thin shell of uniform thickness. An explicit expression for the first-order scattered field in the form of a surface integral is evaluated approximately by the method of stationary phase. Edge effects are ignored. Physical aspects of the solution are discussed and compared with previous results.
C. J. Bouwkamp (Eindhoven).

Honerjäger, Richard. Zur Theorie der elektromagnetischen Strahlung in Metallrohren. Z. Physik **128**, 72–78 (1950).

Any electromagnetic wave in a waveguide may be expressed as the sum of a number of characteristic waves in the admissible *TE* and *TM* modes. The author considers the propagation of waves in an infinite cylindrical guide of arbitrary cross-section when a current sheet of given density, electric and magnetic moments, is introduced at the cross-section $z=0$. Boundary conditions across the sheet are used to determine the coefficients in the expansions of the resultant field in series of admissible modes. The case of an electric or magnetic dipole (either transverse or longitudinal) at the origin is obtained from the general solution by a limiting process. The total electromagnetic power flow through a cross-section of the guide is also evaluated.
M. C. Gray (Murray Hill, N. J.).

Phillips, R. S. The electromagnetic field produced by a helix. Quart. Appl. Math. **8**, 229–246 (1950).

Let $\exp(i\beta z-i\omega t)$ denote the electric-current flow in the helix $\rho=a$, $z=\alpha\varphi$, where z, ρ, φ are cylindrical coordinates. The author calculates the corresponding electromagnetic field from the Hertzian vector Π. The cylindrical components of Π involve integrals of the type

$$\int_{-\infty}^{\infty} e^{i\beta\zeta}(e^{ikR}/R)d\zeta, \qquad R^2=\zeta^2+a^2+\rho^2-2a\rho\cos(\varphi-\zeta/\alpha).$$

This integral is discussed thoroughly and evaluated in terms of an infinite series of Bessel-function products; the convergence of this series is treated in detail. The author's aim is to link the theoretical electromagnetic field of the infinitely long and infinitely thin helical wire with the physical helix in the traveling-wave tube. In the physical problem, the tangential electric field on the surface and in the direction of the (perfectly) conducting helix must be zero. Accordingly, the author determines that value $\beta=\beta_0$ of the propagation constant which makes the electric field "essentially" zero on the helix, in the sense that $\lim_{\rho\to a} E_h(\beta_0)/E_h(\beta)=0$, $\beta^2\neq\beta_0^2$. As may be expected, $\beta_0^2=(k/\alpha)^2(a^2+\alpha^2)$, so that the phase velocity of the traveling wave along the helix must be equal to that in free space. *C. J. Bouwkamp* (Eindhoven).

Ruch, Ernst. Der Einfluss einer Blende in Rohren auf das Feld einer einfallenden elektromagnetischen Welle. Ann. Physik (6) **7**, 248–272 (1950).

Insertion of a diaphragm at any cross-section of a cylindrical waveguide in which a *TE* or *TM* wave is traveling generates reflected and transmitted waves which may be a combination of any number of *TE* and *TM* waves. It is assumed that the diaphragm, or iris, consists of an infinitely thin sheet of perfectly conducting material with a hole of arbitrary dimensions, the outer boundary of the iris coinciding with the guide boundary. When the secondary fields are expanded in series of characteristic functions associated with each mode, the application of the boundary conditions over the surface of the iris leads to an infinite set of algebraic equations in the expansion coefficients. A matrix solution of these equations is discussed in general terms, while a least-squares method may be used if only a finite number of equations is retained. The case of a $TE_{1,0}$ wave in a rectangular waveguide, with an iris whose hole consists of a strip parallel to one edge of the boundary, is discussed in detail, and approximate solutions obtained.
M. C. Gray (Murray Hill, N. J.).

Severin, Hans. Der Schlitzstrahler, ein magnetischer Dipol für Zentimeterwellen. Z. Physik **128**, 108–119 (1950).

When an electromagnetic wave is incident on a small slot in a thin perfectly conducting plate the radiation field from the slot is the same as that from a magnetic dipole whose axis is in the direction of the incident H-vector. A complementary diffraction problem is that of a plane wave in free space incident on a small perfectly conducting rod of the same dimensions as the slot. The analogy is used to determine the radiation field from a slot in the shape of an ellipse, whose major axis is either parallel or perpendicular to H. Satisfactory agreement with experimental measurements was obtained as long as the largest slot dimension was less than $\lambda/3$. *M. C. Gray* (Murray Hill, N. J.).

Djakov, E., und Christov, Chr. Verteilung des elektrischen Potentials in Schlitzanodenmagnetronen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **39**, 95–131 (1943). (Bulgarian. German summary)

Christov, Chr. Sur le mécanisme des oscillations électroniques dans le magnétron à anode fendue. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **43**, 17–42 (1947). (Bulgarian. French summary)
[Volume number misprinted **42** on title page.]

Estrin, Gerald. The effective permeability of an array of thin conducting disks. J. Appl. Phys. **21**, 667–670 (1950).

A three-dimensional array of thin conducting circular disks has found special application as an artificial refractive medium at microwave frequencies. The dielectric tensor of this medium has been calculated by W. E. Kock [Bell System Tech. J. **27**, 58–82 (1948)]. Similarly, the magnetic coefficients are evaluated by the author. For that purpose he calculates the steady-state current distribution induced in the disk by the alternating magnetic field (which is perpendicular to the disk). The method of calculation and the author's result are not new [cf. Jouguet, Courants de Foucault et fours à induction, Gauthier-Villars, Paris, 1944, chapter IV; C. R. Acad. Sci. Paris **216**, 523–524 (1943); these Rev. **5**, 220]. The author emphasizes a missing factor 2 in Jouguet's work. [There seems to be no discrepancy at all, in that Jouguet considers the current on one face of the disk only, while the author evaluates the total current, which is twice as large.] *C. J. Bouwkamp* (Eindhoven).

Snow, Chester. Potential problems and capacitance for a conductor bounded by two intersecting spheres. J. Research Nat. Bur. Standards **43**, 377–407 (1949).

This paper is concerned with the electrostatic field outside a body bounded by two intersecting spheres. The spheres are described by the radius a of one of them, the angle θ subtended by the radius of the circle of intersection at the

centre of the sphere of radius a, and the angle ω of intersection of the two spheres; V is the potential, α, β are toroidal coordinates ($0 \leqq \alpha \leqq \omega$, $\beta \geqq 0$), and $x = \cosh \beta$. A general solution of Laplace's equation may be written as

$$\{\cosh \beta - \cos (\alpha - \theta)\}^{-\frac{1}{2}} V(\alpha, \beta) = W(\alpha, \beta)$$
$$= (\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \nu P_{\nu-\frac{1}{2}}(x) \csc \nu\omega \{F_1(\nu) \sin \nu(\omega - \alpha) + F_2(\nu) \sin \nu\alpha\} d\nu.$$

The functions F_1 and F_2 are determined by the boundary values of the potential on the two spheres $\alpha = 0$ and $\alpha = \omega$. The author discusses conditions under which a given function $f(x)$ can be represented in the form $\pi i \int_{c-i\infty}^{c+i\infty} \nu P_{\nu-\frac{1}{2}}(x) F(\nu) d\nu$. [He does not refer to, or compare his results with, Fock, C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 253–256 (1943); these Rev. **5**, 181.] An alternative approach makes use of the Green's function: this is constructed in terms of a fundamental function $S_\omega(\alpha, \beta, \beta')$ which is defined as an integral and whose properties are investigated. With this set-up the author discusses various potential problems and computes the capacitance of a conductor of the assumed shape. If ω/π is rational, the formulas simplify to a considerable extent. In this case the method of images may be used successfully. *A. Erdélyi* (Pasadena, Calif.).

Eigenson, L. S. Some linear and plane problems of a stationary potential field with variable coefficient of conductivity. Doklady Akad. Nauk SSSR (N.S.) **74**, 53–54 (1950). (Russian)

Let the conductivity $\lambda(t)$ be a function of the potential t. The boundary contour contains two equipotential lines, $t = t_1$, $t = t_2$, which can be mapped from the (x, y)-plane on a (ϕ, ψ)-plane as the lines $\psi = \psi_1$, $\psi = \psi_2$. The analysis shows that the form of the equipotentials is the same whether $\lambda = \lambda(t)$ or $\lambda =$ a constant, though specific potentials may change. A formula for the flux is also derived.

R. E. Gaskell (Ames, Iowa).

Grosskopf, Jürgen. Zur Theorie der geraden Antenne. Arch. Elektr. Übertragung **4**, 175–180 (1950).

The author suggests a modification of the Hallén method for center-driven antennas which uses only the first approximation to the solution of Hallén's integral equation. Instead of an iteration process based on an initial sinusoidal distribution the assumed current is modified in the gap region by introducing a correction current corresponding to a small attenuation factor in the propagation constant. The correction current is sinusoidal, with a maximum value at the center of the gap, and is negligible except in the gap neighborhood. This device leads to a finite value for the input impedance of a half-wavelength antenna, and gives approximate formulas for the anti-resonant impedance and the line shortening at resonance in good agreement with experimental values. *M. C. Gray* (Murray Hill, N. J.).

Gorn, Saul. A graphic method of matching impedances for bandwidth. Army Air Forces, Air Materiel Command, Wright Field, Dayton, Ohio. Mem. Rep. no. **TSELR-158.** 55 pp.+23 tables (13 graphs) (1947).

The problem considered is that of matching a given antenna impedance, which varies with frequency, to a fixed impedance, the characteristic impedance Z_0' of the lossless transmission line which feeds the antenna, by means of a matching section, such that a satisfactory match is obtained over as wide a frequency band as possible. The criterion used to evaluate the "goodness of match" is that the antenna impedance should remain within the 2:1 standing wave circle, $C_{1/2}$ about Z_0' for the frequency band considered. Graphical methods are described for obtaining optimum matching sections of different types, such as shunt and series transmission lines and passive nondissipative four-terminal networks. Problems involving only one or two variable parameters are discussed in general terms, while some specific examples are worked out in detail. The graphical constructions are essentially simple, especially when bipolar paper is available, but they are very difficult to summarize. However, the detailed examples should make it easy to apply the method to any given problem.

M. C. Gray (Murray Hill, N. J.).

Bott, R., and Duffin, R. J. Impedance synthesis without use of transformers. J. Appl. Phys. **20**, 816 (1949).

If a rational function $Z(s)$ of the complex variable s (a) is real for real s and (b) has a positive real part when the real part of s is positive, then $Z(s)$ is a positive real function (prf). Suppose $Z(s)$ has no zeros or poles on the imaginary axis. Let there exist an $\omega > 0$ such that $Z(i\omega)$ is purely imaginary, and assume that $Z(i\omega) = i\omega L$, $L > 0$. Then $Z(s)$ can be written

$$Z(s) = [1/Z_1(s) + Cs]^{-1} + [1/Ls + 1/Z_2(s)]^{-1},$$

where C is a positive constant and $Z_1(s)$ and $Z_2(s)$ are both prf. It is well known that a prf represents the driving point impedance of a lumped, linear, passive network. Here, $Z(s)$ is the impedance of two networks in series. The first consists of the impedance Z_1 in parallel with the capacitance C, and the second consists of Z_2 in parallel with the inductance L. If $Z(i\omega) = -i\omega L$, then $Z(s)$ is the impedance of two networks in parallel. It is shown that the functions Z_1 and Z_2 are of such character that by means of standard network synthesis procedures their complexity may be reduced without the use of transformers. These processes can be repeated again and again until the entire network is obtained. This proof of the realizability of the driving point impedance without the use of transformers is one of the most interesting developments in network theory in recent years.

R. Kahal (St. Louis, Mo.).

Nijenhuis, W. Impedance synthesis distributing available loss in the reactance elements. Philips Research Rep. **5**, 288–302 (1950).

A method is described for the synthesis of a driving point impedance constructed with a finite number of lumped, linear, and lossy inductances and capacitances. The driving point impedance $Z(p)$ [p = complex frequency variable] of such a network is a rational positive real function, and conversely. If $\Re[Z] \neq 0$ on $\Re[p] = 0$, then there exists an $a > 0$ such that $Z(p + a) = Z(\omega)$ is also a positive real function. Thus the synthesis of $Z(\omega)$ corresponds to a synthesis of $Z(p)$ by a network in which each inductance L is accompanied by a series resistance aL, and each capacitance C by a parallel conductance aC. The positive constant a is chosen so that $\Re[Z(\omega)] = 0$ for $\omega = \pm i\lambda_0$. If $Z(\omega)$ has a pole or a zero at $\omega = \pm i\lambda_0$, then the pole or zero is removed by the standard Brune method. If both $Z(\pm i\lambda_0)$ and its reciprocal are different from zero, Brune's method leads to unrealizable networks as, in general, it calls for negative resistance. However, the recently discovered procedure of Bott and Duffin [see the preceding review] may then be invoked to continue the synthesis, and by repeating these steps until the original function $Z(p)$ is exhausted, a physical network may be realized. A short discussion of some possible

modifications of this method that provide losses simulating dissipation varying with frequency is included. Some numerical examples are given. A somewhat different approach to this problem was taken in earlier investigations by Nai-Ta Ming [Sci. Rep. Nat. Tsing Hua Univ. Ser. A. **5**, 350–377 (1949); Arch. Elektrotechnik **39**, 359–387, 452–471 (1949); these Rev. **12**, 148, 149] before the Bott and Duffin results were available. *R. Kahal* (St. Louis, Mo.).

Ghosh, Chandrasekhar. Generalized impedance circle diagrams in the analysis of coupled networks. Indian J. Phys. **24**, 223–231 (1950).

As is well known, the complete specification of a two terminal-pair network requires three parameters. Given three such parameters, an equivalent T circuit representation can always be found, although the branches individually may not be physically realizable. Nevertheless, this device may be useful for purposes of analysis. By considering the complete circuit as an equivalent T section it is shown in this paper that, when any general two terminal-pair network is terminated with a variable complex impedance, the locus of the input impedance at the driving end is a circle in the complex plane. Some examples are shown, and the construction of impedance circle diagrams for these cases is given. *R. Kahal* (St. Louis, Mo.).

Hübner, W. Zur Anwendung der Vierpoltheorie auf die Maxwellschen Gleichungen. Analogiebetrachtungen im Hinblick auf optische Probleme. Optik **7**, 128–146 (1950).

The general techniques of the analysis of lumped element four-pole networks is applied to the two conductor transmission line. From the telegraph equations of the transmission line certain quantities, such as characteristic impedance, propagation constant, and reflection factor, may be defined by analogy with the equations of the four-pole network. If the electric and magnetic field intensities of a plane polarized electromagnetic wave are regarded as the analogues of voltage and current on the transmission line, then a duality is evident between the telegraph equations and Maxwell's equations. A solution of the former yields a solution of the latter when the concepts of the impedance and propagation constant of the medium, and the reflection factor at discontinuities of the medium, are properly defined in terms of the physical properties of the medium. The results may be used to obtain the solution of certain problems in optics, such as reflections from lenses and mirrors, and the calculation of optical band-pass filters. *R. Kahal* (St. Louis, Mo.).

Statistical Mechanics

***Koppe, Heinz. Die Grundlagen der statistischen Mechanik.** S. Hirzel Verlag, Leipzig, 1949. vi+79 pp. DM 3.50.

de Boer, J. Molecular distribution and equation of state of gases. Reports on Progress in Physics **12**, 305–374 (1949).

Casimir, H. B. G. Some aspects of Onsager's theory of reciprocal relations in irreversible processes. Nuovo Cimento (9) **6**, Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 227–231 (1949).

Robb, W. L., and Drickamer, H. G. Transport properties of dense media. I. Thermal diffusion in isotopic mixtures of gases. J. Chem. Phys. **18**, 1380–1382 (1950).

The author develops an expression for the thermal diffusion ratio α, which can be evaluated from the equation of state, and which qualitatively reproduces experimental values in the critical region for ethane-xenon. It is indicated that the effect of pressure on α cannot be attributed to selective clustering. *C. C. Torrance* (Annapolis, Md.).

Band, William. Condensation phenomena in a clustering Bose-Einstein gas. Physical Rev. (2) **79**, 871–876 (1950).

The theory of imperfect gases (gases in which forces exist between atoms) can be treated approximately as a problem in association. That is, at a given temperature volume and pressure the gas can be assumed to be made up as a mixture of single molecules and clusters of $2, 3, 4, \cdots$ atoms. A cluster of n atoms is postulated to have an energy V_n associated with it. By statistical mechanics one can find the distribution of clusters of various sizes. By investigating the change in distribution with volume per molecule one can develop a theory of condensation. In this paper this method is used to study the condensation of an Einstein-Bose gas with intermolecular forces. The quantum character of the Einstein-Bose gas necessitates some changes in the usual mathematical techniques. The most interesting result is that according to the above model an imperfect Einstein-Bose gas cannot undergo the characteristic λ-transition, for the gas has already condensed at the critical temperature of the λ-transition. *E. Montroll* (College Park, Md.).

Cheng, Kai-Chia. A new method for determining the radial distribution function. Proc. Phys. Soc. Sect. A. **63**, 1028–1036 (1950).

Green's approximate integral equation for the radial distribution function of a liquid is investigated. This equation is nonlinear and cannot be solved exactly. The author suggests, however, a method of successive approximation which should converge (under conditions partially investigated) to the exact solution. This is essentially a symbolic method which makes use of the form of the integral operator to transform the integral equation into a differential equation of infinite order. Then, taking the first few terms, this gives an ordinary differential equation which in turn is solved by iteration. The method may be expected to give reasonable results at high temperatures. The special case of nitrogen is carried through in detail using a Lennard-Jones potential, and some agreement with experiment is found. *J. M. Luttinger* (Madison, Wis.).

AUTHOR INDEX

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